ES 331 Probability and Random Processes

SHANMUGA

Based on the book
Intuitive Proabability and Random Processes using MATLAB
by Steven Kay

Contents

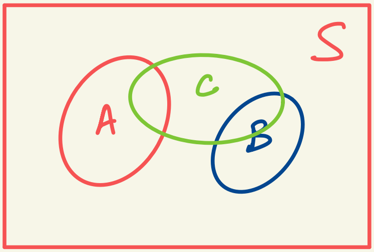
Review of Set Theory

Universal set S

Any subset of S, $A \subset S$

eg: $S = \{0, 1, 2, \ldots\}$

$A = \{0, 2, 4, \ldots\}$



1. $\Phi = S^c$

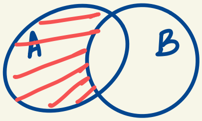
2. $A \cup B = B \cup A$, $A \cap B = B \cap A$ commutative

3. $A \cup (B \cup C) = (A \cup B) \cup C$
   $A \cap (B \cap C) = (A \cap B) \cap C$ Associative

4. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ Distributive

5. $A - B = A \cap B^c$

$= A - (A \cap B)$



6. $(A \cup B)^c = A^c \cap B^c$

7. $(A \cap B)^c = A^c \cup B^c$

8. $A \cup S = S$

9. $A \cap S = A$

10. $A \cup A^c = S$

11. $A \cap A^c = \Phi$

12. $A \cup \Phi = A$

13. $A \cap \Phi = \Phi$

Element and Simpleton Set

eg.

$S = \{0, 1, 2, \ldots\}$

$0 \in S$

$1 \in S$

Elements

$\{0\}, \{1\}, \ldots \subset S$

Size of Set

1. Countably finite set

eg: $A = \{2, 4, 6, 8\}$ $\quad$ $\text{card}(A) = 4$

2. Countably infinite set

eg. $A = \{0, 1, 2, \ldots\}$ $\quad$ $\text{card}(A) = \infty$

3. Uncountably infinite set

eg: $B = \{x : x \in \mathbb{R},\ 0 \le x \le 1\}$

Countable Sets

eg: a) Integers

b) Rational Numbers $\mathbb{Q}$

Uncountable sets (infinite)

eg: a) Real Numbers or its subset

$$\mathbb{R} = Q \cup P$$

Real Rational Irrational

Probability Space

	Set Theory		Probability Space	
1.	Universe	1.	Sample space	S
2.	Subset	2.	Event	$E \subseteq S$
3.	Element	3.	Sample Point or Outcomes	$s \in S$
4.	Simpleton set	4.	Simple event	$\{s\} \subset S$
5.	Null set	5.	Impossible Event	Φ
6.	Disjoint sets	6.	Mutually Exclusive events	$A \cap B = \Phi$

Axioms of Probability

1. Probability of Sample space is 1.

$$P[S] = 1$$

2. Probability of any event is non-negative.

$$P[E] \geq 0 \qquad E \subseteq S$$

3. If A, B events are mutually exclusive,

$$P[A \cup B] = P[A] + P[B]$$

$$\downarrow$$

$$P[A \cap B] = P[\Phi]$$
$$= 0$$

Corollary 1

$$0 \leq P[A] \leq 1$$

Proof

$$A1 \Rightarrow P[S] = 1$$

$$P[A \cup A^c] = 1 \qquad P[A \cap A^c] = P[\Phi] = 0$$

$$A3 \Rightarrow P[A] + P[A^c] = 1$$

$$P[A] = 1 - P[A^c]$$

from A2

$$P[A^c] \geq 0$$

$$\Rightarrow P[A] \leq 1$$

Corollary 2

$$P[\Phi] = 0$$

Proof

A1 $\Rightarrow$ $P[S] = 1$

$$P[\Phi] = P[S^c]$$

$$= 1 - P[S]$$

$$= 1 - 1$$

$$= 0$$

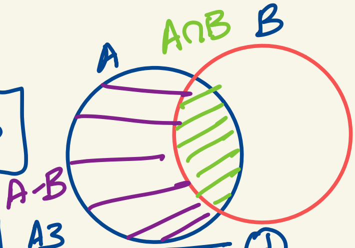
Corollary 3

For any events A and B

$$P[A \cup B] = P[A] + P[B] - P[A \cap B]$$

Proof

$$P[A \cup B] = P[(A-B) \cup B]$$



$$= P[A-B] + P[B] \quad A3 \quad \text{———} ①$$

$$P[A] = P[(A-B) \cup (A \cap B)]$$

$$= P[A-B] + P[A \cap B] \quad A3$$

$$P[A-B] = P[A] - P[A \cap B] \quad \text{———} ②$$

Sub. ② in ①

$$P[A \cup B] = P[A] + P[B] - P[A \cap B]$$

Sample Space

a) Discrete (Countably finite or infinite set)

b) Continuous (Uncountably infinite)

a) Discrete SS

- Assign probability to every sample point.

eg. Toss a die (fair)

$$S = \{\underset{s_1}{1}, \underset{s_2}{2}, \underset{s_3}{3}, \underset{s_4}{4}, \underset{s_5}{5}, \underset{s_6}{6}\}$$

$$P[s_i] = \frac{1}{6} \quad s_i \in S$$

eg. Toss a coin

$$S = \{H, T\}$$

$$P[H] = p \,, \quad P[T] = 1 - p$$

eg. Red, Black balls Combinatorics

Total Balls = N

Red Balls = N_R

Black Balls = N_B $\quad N_R + N_B = N$

$$p_R = \frac{N_R}{N} \qquad p_B = \frac{N_B}{N} = 1 - p_R$$

Draw M balls with replacement

Arrange k Red Balls out of M

Permutations

$$MP_k = M \cdot (M-1) \cdot (M-2) \ldots M-(k-1) \quad \text{Tuple}$$

$$= \frac{M!}{(M-k)!}$$

Combinations

$$\binom{M}{k} = MC_k = \frac{M!}{(M-k)!\, k!} \quad \text{Set}$$

Binomial Law

$$P\left[k \text{ red balls out of } M \text{ balls}\right] = \binom{M}{k} p_R^k (1-p_R)^{M-k}$$

$k = 0, 1, 2, \ldots, M$

$M << N$, Binomial law is valid for M balls drawn without replacement.

Multinomial Law

Total N balls of n colors

$p_1, p_2, \dots, p_n \rightarrow$ Probabilities of different colored balls

M balls drawn

$k_1, k_2, \dots, k_n$ $\qquad k_1 + k_2 + \dots + k_n = M$

$$P[\;] = \frac{M!}{k_1!\, k_2! \dots k_n!} p_1^{k_1} p_2^{k_2} \dots p_n^{k_n}$$

Multinomial Law

$n \rightarrow$ # of colors

Binomial Law

$$\frac{M!}{\underset{\text{red}}{k!}\ \underset{\text{Black}}{(M-k)!}} \, p_R^{k} \, p_B^{(M-k)}$$

b) Continuous Sample Space

$$S = \{ x : -\tfrac{1}{2} \le x \le \tfrac{1}{2} \}$$

$P[S] = 1$ → violated if we assign finite probability to every sample point.

$$P[A = x_0] = 0$$

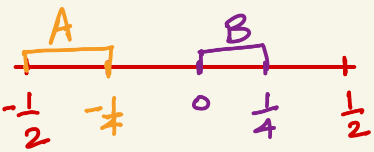
$$-\tfrac{1}{2} \le x_0 \le \tfrac{1}{2}$$

Define probabilities for intervals

$$P[a \le x \le b] = b - a$$

$$P[A] = P[-\tfrac{1}{2} \le x \le -\tfrac{1}{4}] = \frac{1}{4}$$

$$P[B] = P[0 \le x \le \tfrac{1}{4}] = \frac{1}{4}$$



$$P[A \cup B] = P[A] + P[B]$$
$$= \frac{1}{4} + \frac{1}{4}$$
$$= \frac{1}{2}$$

$$P[a \le x \le b] = P[a < x \le b] = P[a \le x < b] = P[a < x < b]$$

as $P[x = a] = 0$

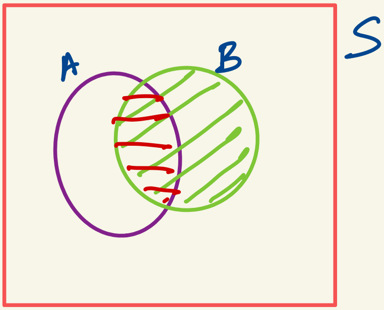
$P[x = b] = 0$

$$P[A] = \frac{Area(A)}{Area(S)}$$

Conditional Probability

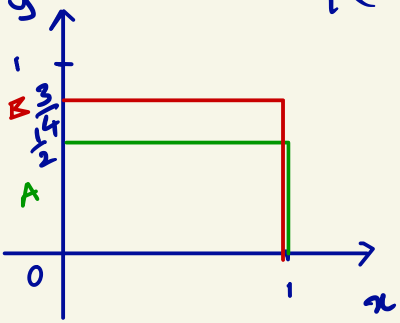
A, B be two events. $P[A]$, $P[B]$

$P[A|B]$ or $P[B/A]$



$$P[A/B] = \frac{P[A\cap B]}{P[B]} \qquad P[B/A] = \frac{P[A\cap B]}{P[A]}$$

$$S = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 1\}$$



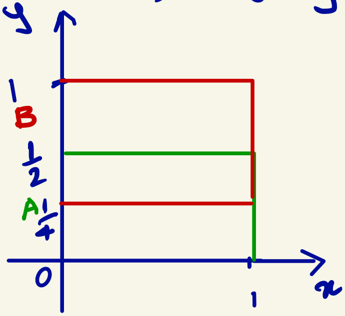
$P[A] = \frac{1}{2}$

$P[B] = \frac{3}{4}$

$P[A/B] = \frac{2}{3} = \frac{P[A \cap B]}{P[B]} = \frac{1}{2} / \frac{3}{4}$

a)

$P[A|B] > P[A]$



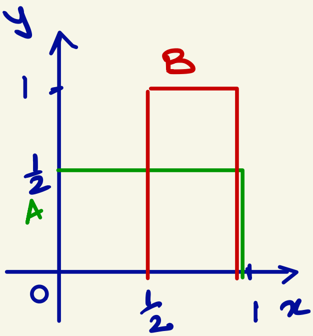
$P[A] = \frac{1}{2}$

$P[B] = \frac{3}{4}$

$P[A/B] = \frac{1}{3}$

b)

$P[A|B] < P[A]$



$P[A] = \frac{1}{2}$

$P[B] = \frac{1}{2}$

$P[A/B] = \frac{1}{2}$

c)

$P[A/B] = P[A]$

A, B are statistically independent

iff $P[A/B] = P[A]$ and $P[B/A] = P[B]$

$$P[A|B] = \frac{P[A \cap B]}{P[B]} \Rightarrow P[A \cap B] = P[A/B]\, P[B]$$

A, B independent

$$\underbrace{P[A \cap B]}_{\text{Joint}} = \underbrace{P[A]\ P[B]}_{\text{Marginal}}$$

A, B → any two events

Stat. independent iff $P[A \cap B] = P[A] \cdot P[B]$

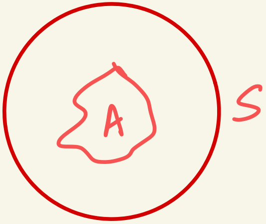
Mutually exclusive iff $P[A \cap B] = 0$

$$P[A|B] = \frac{P[A \cap B]}{P[B]}$$

$$P[B/A] = \frac{P[A \cap B]}{P[A]}$$

Corollaries for Conditional Probability from the Axioms

1. $P[S/A] = 1$



Proof

$$P[S/A] = \frac{P[S \cap A]}{P[A]}$$

$$= \frac{P[A]}{P[A]}$$

$$= 1$$

2. $P[A/B] \geq 0$, $P[B] \neq 0$

Proof

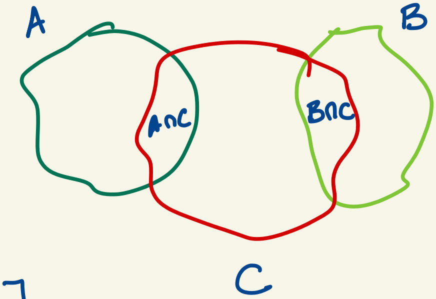
Ⓐ②

$$P[A|B] = \frac{P[A\cap B] \geq 0}{P[B] > 0}$$

$$\geq 0$$

3. $P[A\cup B/C] = P[A/C] + P[B/C]$ if A and B are Mutually Exclusive

$P[A\cap B] = 0$



Proof

$$P[A\cup B/C]$$

$$= \frac{P[(A\cup B)\cap C]}{P[C]}$$

$$= \frac{P[(A\cap C)\cup(B\cap C)]}{P[C]} \longrightarrow A3$$

$$= \frac{P[A\cap C] + P[B\cap C]}{P[C]}$$

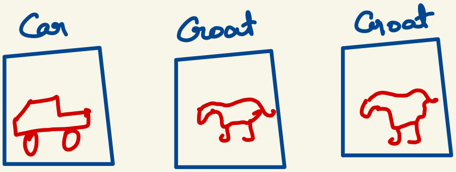
$$= \frac{P[A\cap C]}{P[C]} + \frac{P[B\cap C]}{P[C]}$$

$$= P[A/C] + P[B/C]$$

eg. Monty Hall's Problem

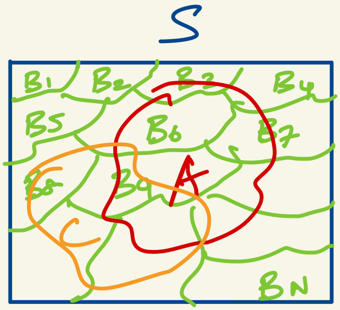
P[winning a car]

No switch $\frac{1}{3}$ → Switch $\frac{2}{3}$



$\frac{1}{3}$

Law of Total Probability



$S \rightarrow$ Sample space

$\{B_i\}_{i=1}^{N} \rightarrow$ Mutually Exclusive and Exhaustive events

M.E. $B_i \cap B_j = \Phi, \; i \neq j$

Exh. $\bigcup_{i=1}^{N} B_i = S$

$P[B_i \cap B_j] = 0, \quad \sum_{i=1}^{N} P[B_i] = 1$

LTP $P[A] = \sum_{i=1}^{N} P[A/B_i] \, P[B_i]$

Proof

$$P[A] = P[A \cap S]$$

$$= P\left[A \cap \left(\bigcup_{i=1}^{N} B_i\right)\right]$$

$$= P\left[\bigcup_{i=1}^{N} (A \cap B_i)\right] \qquad A3$$

$$= \sum_{i=1}^{N} P[A \cap B_i]$$

$$P[A] = \sum_{i=1}^{N} P[A/B_i]\, P[B_i]$$

Baye's Theorem

$$P[A \cap B] = P[A/B]\, P[B]$$

$$P[A \cap B] = P[B/A]\, P[A]$$

$$P[A/B]\, P[B] = P[B/A]\, P[A]$$

$$P[B/A] = \frac{P[A/B]\, P[B]}{P[A]}$$

$$P[B_k/A] = \frac{P[A/B_k]\, P[B_k]}{P[A]}$$

TLP

Posterior prob. $P[B_k/A] = \dfrac{P[A/B_k]\, P[B_k]}{\sum_{i=1}^{N} P[A/B_i]\, P[B_i]}$

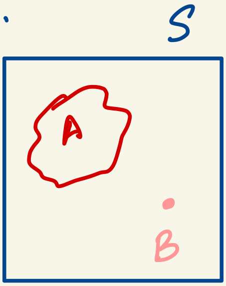
Stat. independent iff $P[A\cap B] = P[A]\cdot P[B]$

Mutually exclusive iff $P[A\cap B] = 0$

1. $A, B \rightarrow$ M.E. & Ind.

Cont.

eg.



$P[A] > 0$

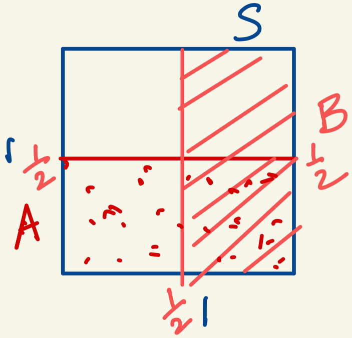
$P[B] = 0$

M.E. as $P[A\cap B] = 0$ & Ind. as $P[A]\cdot P[B] = 0$

$= P[A\cap B]$

2. $A, B \rightarrow$ Not M.E. & Ind.

eg.



$P[A] = \frac{1}{2}$ $P[B] = \frac{1}{2}$

$P[A\cap B] = \frac{1}{4}$ Not M.E.

$P[A]\cdot P[B] = \frac{1}{4}$ Ind.

$= P[A\cap B]$

3. $A, B \rightarrow$ Not M.E. & Not Ind.

eg. $S = \{1,2,3,4,5,6\}$ $A = \{1,2,4,6\}$

$P[A] = \frac{4}{6}$ $P[B] = \frac{4}{6}$ $B = \{2,4,5,6\}$

$P[A \cap B] = \frac{3}{6}$ → Not . M.E.

$P[A] . P[B] = \frac{16}{36} \neq P[A \cap B]$ Not Ind.

4. A, B → M.E. & Not Ind.

<u>eg.</u> $S = \{H, T\}$

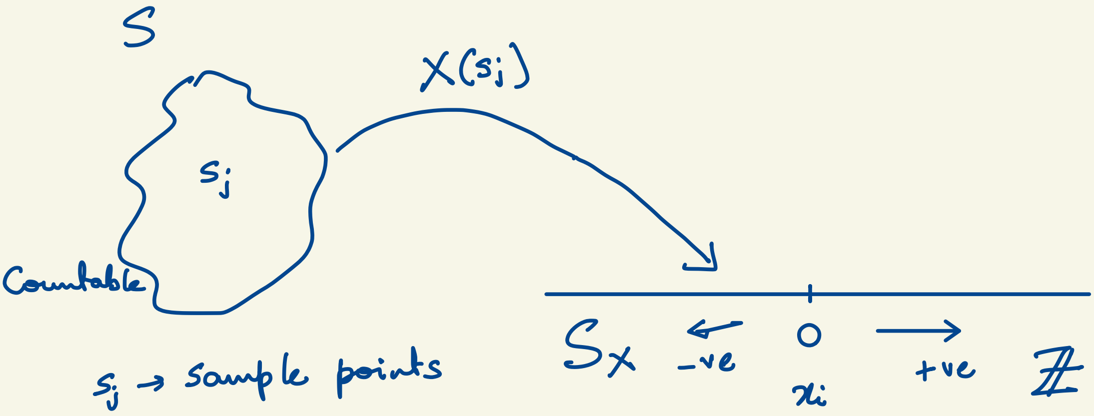
$A = \{H\}$, $B = \{T\}$

$P[A] = \frac{1}{2}$, $P[B] = \frac{1}{2}$

$A \cap B = \Phi$ $P[A \cap B] = 0$ M.E.

$P[A] . P[B] = \frac{1}{4} \neq P[A \cap B]$ Not. Ind.

Discrete Random Variable



s_j → sample points

Integer
or
~~Rational~~

Not done commonly

One-to-one

$$x_i = X(s_j)$$

Many-to-one

$$x_i = X(s_j) \, , \quad j = 1, 2, \ldots, P$$

$$X(\overset{\text{domain}}{\cancel{s_j}}) = \overset{\text{range}}{x_i}$$

$$\underset{\text{function}}{X} = \underset{\text{range}}{x_i}$$

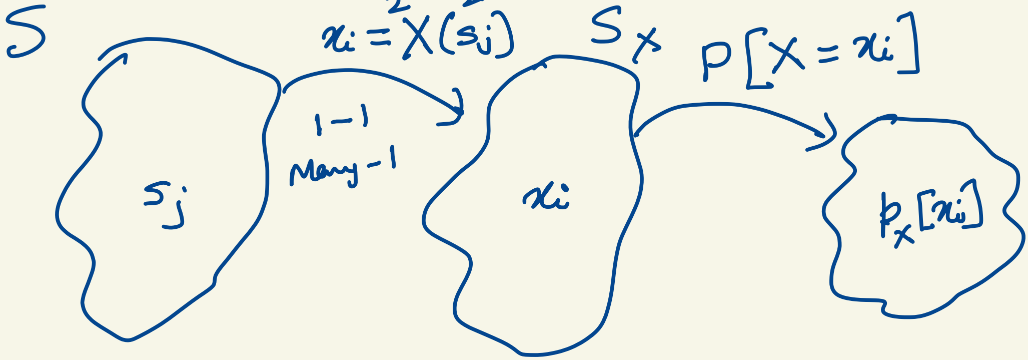
$x_i \in \mathbb{Z}$ Integers

$X \rightarrow$ Discrete Random Variable eg: Coin Toss $S = \{ \underset{\frac{1}{2}}{H}, \underset{\frac{1}{2}}{T} \}$

$$X(H) = 1$$

$$X(T) = 0$$

$$S_X = \{ \underset{\frac{1}{2}}{1}, \underset{\frac{1}{2}}{0} \}$$

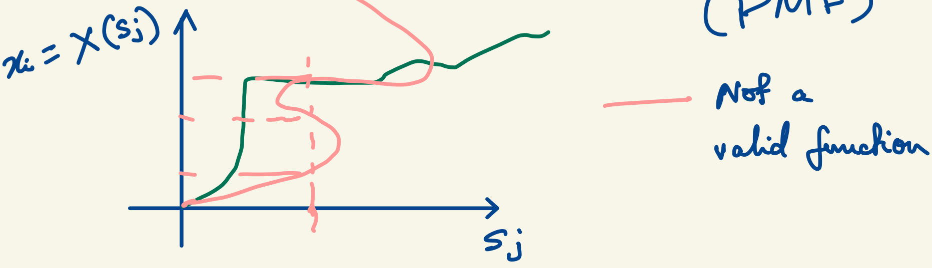


$x_i \in \mathbb{Z}$

$$X = x_i \, , \; i = 1, 2, \ldots, n$$

$$P[X = x_i] = p_X[x_i]$$

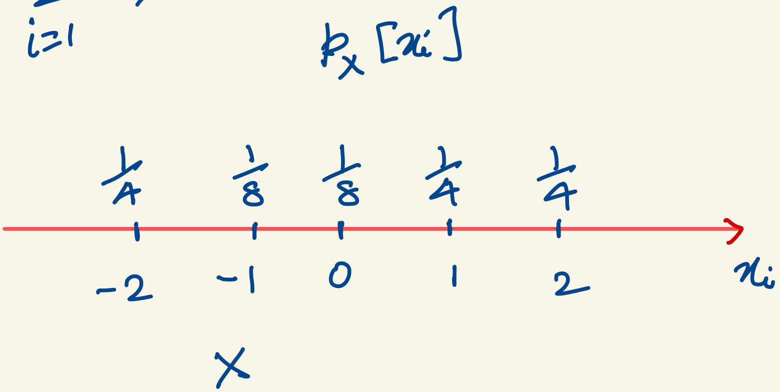
$p_X[x_i]$ ⟶ Probability Mass function (PMF)



From axioms

1. $0 \le p_X[x_i] \le 1$

2. $\sum_{i=1}^{n} p_X[x_i] = 1$



One-to-one mapping

$$p_X[x_i] = P[s_j]$$

$$\equiv P[X(s_j) = x_i] = P[s_j]$$

Many-to-one mapping

$$p_X[x_i] = \sum_{\{j : X(s_j) = x_i\}} P[s_j]$$

Examples of PMF

1. Bernoulli PMF Ber(p)

$$p_X[k] = \begin{cases} p &, \text{ if } \overset{X}{k = 1} \\ (1-p) &, \text{ if } k = 0 \end{cases}$$

eg. (Un)Biased Coin Toss

$X \sim \text{Ber}(p)$ $\sim$ → distributed according to

2. Binomial PMF $X \sim \text{bin}(M, p)$

$$p_X[k] = \binom{M}{k} p^k (1-p)^{M-k}, \quad k = 0, 1, 2, \ldots, M$$

Repeated Bernoulli Trials (k success out of M trials.)

3. Geometric PMF $X \sim \text{geom}(p)$

$$p_X[k] = (1-p)^{k-1} p \qquad k = 1, 2, \ldots$$

First success in k^{th} Bernoulli Trial

4. Poisson PMF $X \sim \text{Pois}(\lambda)$

$$p_X[k] = \frac{e^{-\lambda} \lambda^k}{k!} \qquad k = 0, 1, 2, \ldots$$

$\lambda > 0$ $\lambda = Mp$ $\begin{matrix} M \to \infty \\ p \to 0 \end{matrix}$

when success is rare in M Bernoulli Trials

Binomial → Poisson $\begin{matrix} M \to \infty \\ p \to 0 \end{matrix}$ (Book)

eg: Toss of 2 coins (unbiased)

$S = \{HH, HT, TH, TT\}$

$\frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4} \qquad P[s_j]$

No. of heads

$X = \{0, 1, 2\}$

$P_X[x_i] \quad \frac{1}{4} \quad \frac{1}{2} \quad \frac{1}{4}$

$0 = X(TT)$

$1 = X(HT \text{ or } TH)$

$2 = X(HH)$

$P_X[0] = \frac{1}{4} \rightarrow P[X=0]$

$P_X[1] = \frac{1}{2} \rightarrow P[X=1]$

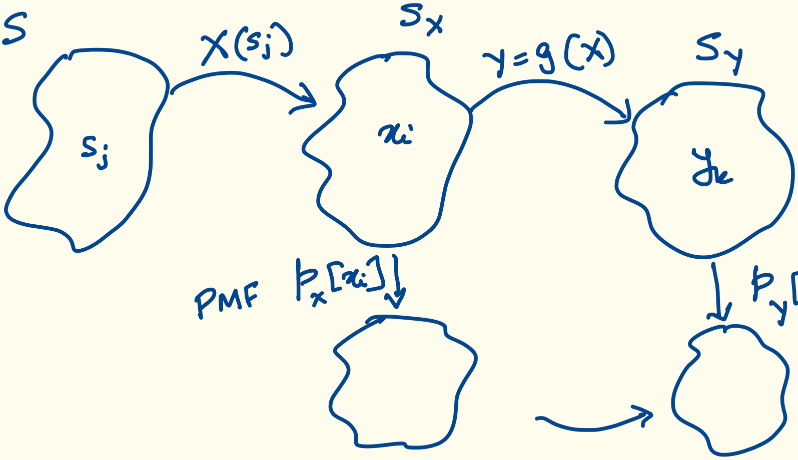
$P_X[2] = \frac{1}{4} \rightarrow P[X=2]$

PMF

1kg of sand

→ Distribute among all values or locations in $\mathbb{Z}$ which are assigned to Discrete Random Variable X

Function of a Discrete Random Variable



$Y \rightarrow$ Discrete R.V

$Y=g(X)$

$$p_X[x_i] = \sum_{\{s_j : x_i = X(s_j)\}} P[s_j] \qquad S \longmapsto S_X$$

$$p_Y[y_k] = \sum_{\{x_i : y_k = g(x_i)\}} p_X[x_i] \qquad S_X \longmapsto S_Y$$

eg.

$S_X = \{0, 1\}$

$p_X[x_i]$ $\frac{1}{2}$ $\frac{1}{2}$

$Y = 2X - 1$

$S_Y = \{-1, +1\}$

$p_Y[y_k]$ $\frac{1}{2}$ $\frac{1}{2}$

eg.

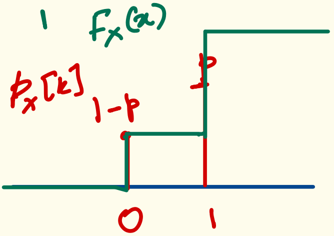
$$S_X = \{-1,\ 0,\ +1\}$$
$$p_X[x_i] \quad \tfrac{1}{4} \quad \tfrac{1}{2} \quad \tfrac{1}{4}$$

$$Y = X^2$$

$$S_Y = \{+1,\ 0\}$$
$$p_Y[y_k] \quad \tfrac{1}{2} \quad \tfrac{1}{2}$$

Cumulative Distribution Function (CDF)

$$F_X(x) = \sum_{\{k\,:\,k \leq x\}} p_X[k] = P[X \leq x]$$



eg. $X \sim \text{Ber}(p)$

$$p_X[k] = \begin{cases} 1-p, & k=0 \\ p, & k=1 \end{cases}$$

$$p_X[k] = F_X(k) - F_X(k^-)$$
$$\underline{\qquad\qquad}$$
$$F_X(k-1)$$

$$F_X(x) = \begin{cases} 0, & x<0 \\ 1-p, & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases}$$

1. $0 \leq F_X(x) \leq 1$

$$F_X(x) = P[X \leq x] \longrightarrow$$ Prob. from axioms proved

2. $F_X(x)$ is monotonically non-decreasing function of x.

$$x_1 \leq x_2$$

$$F_X(x_1) \leq F_X(x_2)$$

3. $F_X(x)$ is right continuous.

$$\lim_{x \to x_0} F_X(x) = F_X(x_0^+)$$

4. Probability in intervals from $F_X(x)$

$$P[a < X \leq b] = F_X(b) - F_X(a)$$

$$P[a \leq X \leq b] = F_X(b) - F_X(a) + P[X = a]$$

$$P[a < X < b] = F_X(b) - F_X(a) - P[X = b]$$

$$P[a \leq X < b] = F_X(b) - F_X(a) - P[X = b] + P[X = a]$$

5. $$\lim_{x \to -\infty} F_X(x) = 0$$

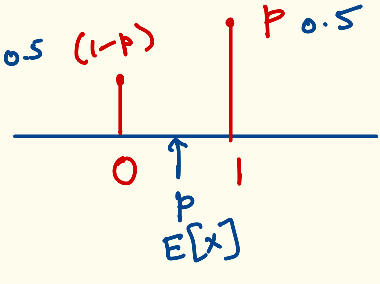
$$\lim_{x \to \infty} F_X(x) = 1 \quad \text{as} \quad \sum_{k=-\infty}^{+\infty} p_X[k] = 1$$

Expectation of a Discrete Random Variable

$$E[X] = \sum_{k=-\infty}^{+\infty} k \, p_X[k]$$

eg. $X \sim \text{Ber}(p)$ $\quad p_X[k] = \begin{cases} 1-p, & k=0 \\ p, & k=1 \end{cases}$

$$E[X] = 0 \cdot (1-p) + 1 \cdot p = p$$



$p = 0.5$

eg. $X \sim \text{Bin}(M, p)$

$$p_X[k] = \binom{M}{k} p^k (1-p)^{M-k}, \quad k = 0, 1, 2, \ldots, M$$

$$E[X] = \sum_{k=0}^{M} \cancel{k} \frac{M!}{\underset{(k-1)!}{\cancel{k!}}\, (M-k)!} p^k (1-p)^{M-k}$$

$$= \sum_{k=0}^{M} \frac{M!}{(k-1)!\,((M-1)-(k-1))!} p^k (1-p)^{(M-1)-(k-1)}$$

$$= Mp \sum_{k=1}^{M} \frac{(M-1)!}{(k-1)!\,((M-1)-(k-1))!} p^{k-1} (1-p)^{(M-1)-(k-1}$$

$$M' = M-1, \quad k' = k-1$$

$$= Mp \underbrace{\sum_{k'=0}^{M'} \frac{M'!}{k'!\,(M'-k')!} p^{k'} (1-p)^{M'-k'}}_{\sum_k p_X[k] = 1} \quad 1$$

$$E[X] = Mp$$

eg. $X \sim \text{Pois}(\lambda)$

$\lambda = Mp$

$M \to \infty$

$p \to 0$

$$p_X[k] = \frac{e^{-\lambda} \lambda^k}{k!}, \quad k = 0, 1, 2, \ldots$$

$$E[X] = \sum_{k=0}^{\infty} k \cdot \frac{e^{-\lambda} \lambda^{k}}{k!}$$

$$= \sum_{k=1}^{\infty} \frac{e^{-\lambda} \lambda^{k-1} \cdot \lambda}{(k-1)!}$$

$$k' = k-1$$

$$= \lambda \underbrace{\sum_{k'=0}^{\infty} \frac{e^{-\lambda} \lambda^{k'}}{k'!}}_{1}$$

$$E[X] = \lambda$$

$E[X]$ exists iff $\displaystyle\sum_{k=-\infty}^{+\infty} |k| \, p_X[k] < \infty$

eg.

$$p_X[k] = \frac{4/\pi^2}{k^2} \qquad k = 1, 2, \ldots$$

$$p_X[k] = 2^{-k} \qquad k = 1, 2, \ldots$$

$E[X]$ does not exist. $\displaystyle\sum_{k=1}^{+\infty} |k| \, 2^{-k} \cancel{< \infty}$

$$E[X] = \sum_{k} k \, p_X[k] \xrightarrow{\text{exist}} \sum_{k} |k| \, p_X[k] < \infty$$

$$Y = g(X)$$

$$E[Y] = E[g(X)] = \sum_{k=-\infty}^{+\infty} g(k)\, p_x[k]$$

{Proof in Book}

$$E[Y] = \sum_{k=-\infty}^{+\infty} k\, p_y[k]$$

eg.

$$S_x = \{-2, -1, 0, 1, 2\}$$

$$p_x[k] \quad \frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{8} \quad \frac{1}{8} \quad \frac{1}{4}$$

$$Y = X^2$$

$$E[Y] = \sum_{k=-2}^{+2} g(k)\, p_x[k] = \sum_{k=-2}^{+2} k^2\, p_x[k]$$

$$= (-2)^2 \cdot \frac{1}{4} + (-1)^2 \cdot \frac{1}{4} + 0^2 \cdot \frac{1}{8} + 1^2 \cdot \frac{1}{8}$$

$$+ 2^2 \cdot \frac{1}{4}$$

$$Y = aX + b$$

$$E[b] = \sum_k b\, p_X[k] = b$$

$$E_Y[Y] = E_X[aX + b] = a E_X[X] + b$$

Relation between Bin(M, p) and Pois(λ)

Binomial

$$p_X[k] = \binom{M}{k} p^k (1-p)^{M-k}, \quad k = 0, 1, 2, \ldots, M$$

Let $Mp = \lambda \Rightarrow p = \frac{\lambda}{M}$ $\quad p \to 0$, $M \to \infty$

$$p_X[k] = \frac{(M)_k}{k!} \left(\frac{\lambda}{M}\right)^k \left(1 - \frac{\lambda}{M}\right)^{M-k} \qquad (M_k) \equiv MP_k$$

$$= \frac{\lambda^k}{k!} \frac{(M)_k}{M^k} \frac{\left(1-\frac{\lambda}{M}\right)^M}{\left(1-\frac{\lambda}{M}\right)^k}$$

as $M \to \infty$ $\quad (M)_k = M(M-1)\ldots(M-(k-1)) \approx M^k$

as $M \to \infty$ $\quad \left(1 - \frac{\lambda}{M}\right)^k \approx 1$

$$p_X[k] = \frac{\lambda^k}{k!} \lim_{M \to \infty} \left(1 - \frac{\lambda}{M}\right)^M \quad \text{———} \ ①$$

$$g(M) = \left(1-\frac{\lambda}{M}\right)^{M}$$

$$\ln g(M) = M \ln\left(1-\frac{\lambda}{M}\right)$$

$$\lim_{M\to\infty} \ln g(M) = \lim_{M\to\infty} M \ln\left(1-\frac{\lambda}{M}\right) = \lim_{M\to\infty} \frac{\ln\left(1-\frac{\lambda}{M}\right)}{\frac{1}{M}}$$

By L'Hopital's rule

$$= \lim_{M\to\infty} \frac{\frac{1}{\left(1-\frac{\lambda}{M}\right)}\left(\frac{\lambda}{\cancel{M^2}}\right)}{-\frac{1}{\cancel{M^2}}}$$

$$= \lim_{M\to\infty} -\frac{\lambda}{1-\frac{\lambda}{M}}$$

$$\lim_{M\to\infty} \ln\left(1-\frac{\lambda}{M}\right)^{M} = -\lambda$$

$\Rightarrow \displaystyle\lim_{M\to\infty} \left(1-\frac{\lambda}{M}\right)^{M} = e^{-\lambda}$ —— ②

② in ①

Poisson PMF $\quad p_X[k] = \dfrac{e^{-\lambda}\lambda^{k}}{k!}, \quad k = 0, 1, 2, \dots$

Minimum Mean Squared Error (MMSE) Estimation

$$E[g(X)] = \sum_{k=-\infty}^{+\infty} g(k)\, p_X[k]$$

$$\sum_{k=-\infty}^{+\infty} (k-b)^2\, p_X[k]$$

$$E[aX+b] = a\,E[X] + b$$

Minimize

$$\underbrace{E}_{\text{Mean}}\Big[\underbrace{(X-\overset{\text{real number}}{b})^2}_{\text{squared}}\Big] = E\left[X^2 - 2bX + b^2\right]$$

$$g(x) = (x-b)^2 \qquad = E[X^2] - 2b\,E[X] + b^2$$

$$b_{opt} = \arg\min_{b}\; E[X^2] - 2b\,E[X] + b^2$$

$$\frac{\partial}{\partial b} E[(X-b)^2] = \frac{\partial}{\partial b}\left[E[X^2] - 2b\,E[X] + b^2\right]$$

$$= 0$$

$$\Rightarrow -2\,E[X] + 2\,b_{opt} = 0$$

$$b_{opt} = E[X] \rightarrow \text{Mean of } X$$

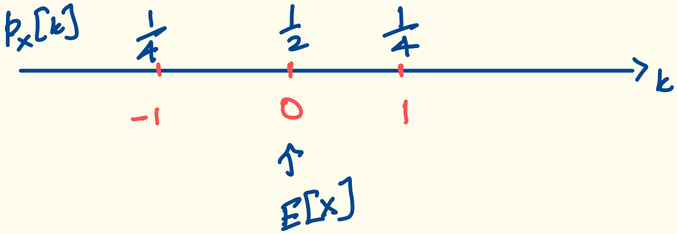
MMSE

$$E\left[(X - b_{opt})^2\right] = E\left[(X - E[X])^2\right]$$

$$= E\left[X^2 - 2\,E[X]\,X + (E[X])^2\right]$$

$$\mathrm{Var}(X) = E\left[(X - E[X])^2\right] = E[X^2] - 2E[X]E[X] + (E[X])^2$$

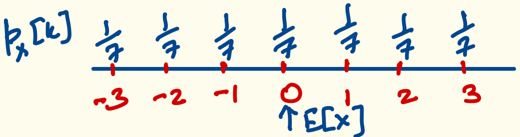
$$= E[X^2] - (E[X])^2$$



eg. Discrete Uniform Random Variable

$$p_x[k] = \begin{cases} \frac{1}{2m+1}, & k = -m, \dots, 0, \dots, m \\ 0, & o/w \end{cases}$$

$m = 3$



$$E[x] = \sum_{k=-m}^{m} \frac{1}{2m+1} k$$

$$= \frac{1}{2m+1} \sum_{k=-m}^{+m} k$$

$$= \frac{1}{2m+1} \cdot 0$$

$$= 0$$

$$Var(x) = \underline{E}\left[(x - E[x])^2\right]$$

$$= E\underset{g(x)}{[x^2]}$$

$$= \sum_{k=-m}^{m} k^2 \frac{1}{2m+1}$$

$$= \frac{1}{2m+1} 2 \sum_{\substack{k=0 \\ \text{or} \\ 1}}^{m} k^2$$

$$= \frac{\overset{1}{\cancel{2}}}{\cancel{2m+1}} \cdot \frac{m(m+1)\cancel{(2m+1)}}{\underset{3}{\cancel{6}}}$$

$$= \frac{m(m+1)}{3}$$

$E[X] = 0$

$$\text{var}(x) = \frac{m(m+1)}{3}$$

as $m\uparrow$, $E[x] = 0$ $\quad$ $\text{var}(x) \uparrow\uparrow$

Moments of Discrete Random Variable X

Moment of X

$$\underset{g(x)}{E[X^r]} = \sum_{k=-\infty}^{+\infty} k^r p_k[k] \rightarrow r^{th} \text{ moment of } X$$

eg. $r = 1$

$E[X] \rightarrow$ Mean

$$E[Y] = E[X - E[X]] = E[X] - E[X] = 0$$

Central Moments of X

$$E[(\underbrace{X - E[X]}_{Y})^r] = \sum_{k=-\infty}^{+\infty} (k - E[X])^r p_X[k] \rightarrow r^{th} \text{ central moment of } X$$

$Y \rightarrow E[Y] = 0$

eg. $r = 2$

$E[(X - E[X])^2] \rightarrow \text{var}(X)$

Characteristic Function $\Phi_X(\omega)$ of R.V. X

$$\Phi_X(\omega) = E\left[e^{j\omega X}\right] = \sum_{k=-\infty}^{+\infty} e^{j\omega k}\, p_X[k]$$

Properties

1. $\Phi_X(\omega)$ always exists for any PMF $p_X[k]$

$$|\Phi_X(\omega)| \le 1$$

Proof

$$|\Phi_X(\omega)| = \left| \sum_{k=\infty}^{+\infty} e^{j\omega k}\, p_X[k] \right| \le \sum_{k=-\infty}^{+\infty} \underbrace{\left|e^{j\omega k}\right|}_{1} \left|p_X[k]\right|$$

$$e^{j\omega k} = \cos \omega k + j \sin \omega k$$

$$= \sum_{k=-\infty}^{+\infty} \left|p_X[k]\right|$$

$$= \sum_{k=-\infty}^{+\infty} p_X[k]$$

$$= 1$$

$$|\Phi_X(\omega)| \le 1$$

2. $\Phi_X(\omega)$ is periodic with period 2π.

$$\Phi_X(\omega+2\pi m) = \Phi_X(\omega) \quad \forall\ m \in \mathbb{Z}$$

Proof

$$\Phi_X(\omega+2\pi m) = E\left[e^{j(\omega+2\pi m)X}\right]$$

$$= \sum_{k=-\infty}^{+\infty} e^{j(\omega+2\pi m)k}\, p_X[k]$$

$$= \sum_{k=-\infty}^{+\infty} e^{j\omega k}\, p_X[k]\, \underbrace{e^{j2\pi mk}}_{1}$$

$$e^{j2\pi mk} = \underbrace{\cos 2\pi mk}_{1} + j \underbrace{\sin 2\pi mk}_{0} \qquad m,k \in \mathbb{Z}$$

$$= 1$$

$$\Phi_X(\omega+2\pi m) = \sum_{k=-\infty}^{+\infty} p_X[k]\, e^{j\omega k}$$

$$= \Phi_X(\omega)$$

3. $p_X[k]$ from $\Phi_X(\omega)$

$$\Phi_X(\omega) = \sum_{k=-\infty}^{+\infty} p_X[k]\, e^{j\omega k} \qquad \text{DTFT} \quad \omega \to -\omega$$

$$p_X[k] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_X(\omega)\, e^{-j\omega k}\, d\omega \qquad \text{IDTFT} \quad \omega \to -\omega$$

Moments from $\Phi_X(\omega)$

$$\frac{d}{d\omega} \Phi_X(\omega) \Big|_{\omega=0} = \frac{d}{d\omega} \sum_{k=-\infty}^{+\infty} e^{j\omega k} p_X[k] \Big|_{\omega=0}$$

$$= \sum_{k=-\infty}^{+\infty} jk \, e^{j\omega k} p_X[k] \Big|_{\omega=0}$$

$$= j \sum_{k=-\infty}^{+\infty} k \, e^{j\omega k} p_X[k] \Big|_{\omega=0}$$

$$= j \sum_{k=-\infty}^{+\infty} k \, p_X[k]$$

$$= j \, E[X]$$

$$E[X] = \frac{1}{j} \frac{d}{d\omega} \Phi_X(\omega) \Big|_{\omega=0}$$

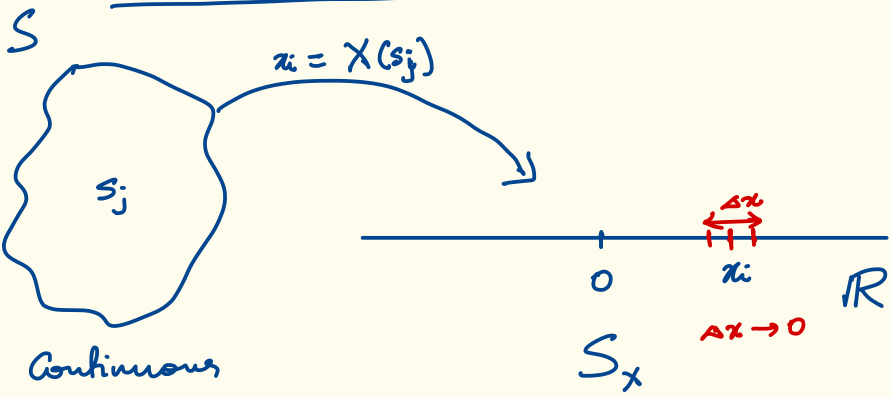
$$\vdots$$

$$E[X^r] = \frac{1}{j^r} \frac{d^r}{d\omega^r} \Phi_X(\omega) \Big|_{\omega=0}$$

$$Y = X - E[X]$$

HW Calculate $\Phi_X(\omega)$ for $Ber(p)$, $Bin(M, p)$, $Pois(\lambda)$, $Geom(p)$

Continuous Random Variable



Continuous
Uncountable
Infinite

$X(\cancel{s_j}) = x_i \qquad x_i \in \mathbb{R}$

Cont $_{r.v}$ $\quad X = x_i \qquad S_x \subseteq \mathbb{R}$

$P[X = x_i] = 0$ (literal)

Definition

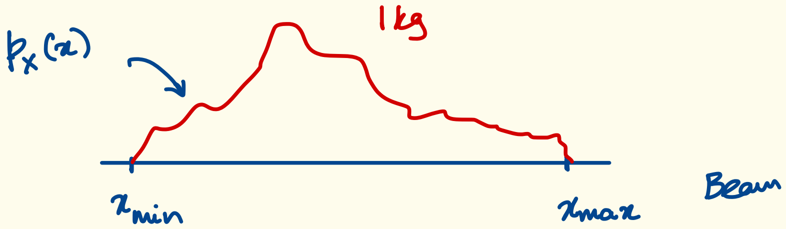
$$p_x(x_i) = P[X = x_i] = \lim_{\Delta x \to 0} \frac{P\left[x_i - \frac{\Delta x}{2} < X \leq x_i + \frac{\Delta x}{2}\right]}{\Delta x}$$

Probability Density Function

$$p_x(x) = P[X = x_i]$$

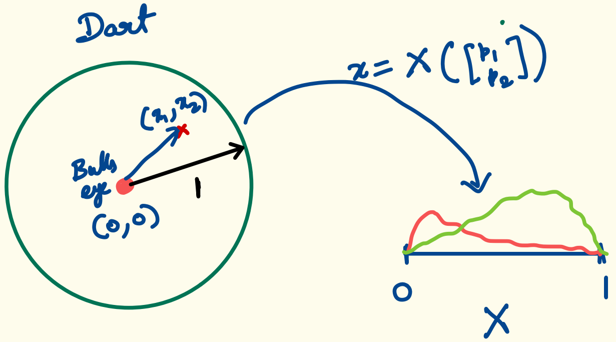
1. $p_x(x) \geq 0$

2. $\int_{-\infty}^{+\infty} p_x(x)\, dx = 1$



X

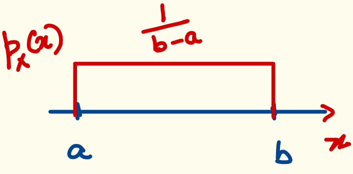
eg: Dart



$$P[a \le X \le b] = P[a < X \le b] = P[a < X < b] = P[a \le X < b]$$

eg: Uniform Random Variable

$$p_X(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & o/w \end{cases}$$



$a = 0, \quad b = 1$

$$\int_a^b \frac{1}{b-a} dx = 1$$

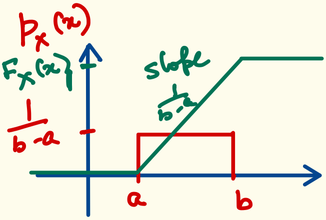
$$p_X(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & o/w \end{cases}$$

Cumulative Distribution Function (CDF)

$$F_X(x) = \int_{-\infty}^{x} p_X(x')\, dx' = P[X \leq x]$$

$$p_X(x) = \frac{d}{dx} F_X(x)$$

1. Uniform

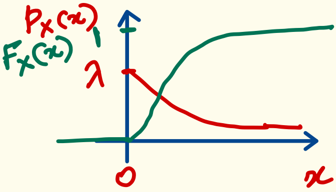


$$p_X(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & o/w \end{cases}$$

$$F_X(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \leq x < b \\ 1, & x \geq b \end{cases}$$

$$\int_a^x \frac{1}{b-a}\, dx' = \frac{1}{b-a}\, x' \Big|_a^x = \frac{x-a}{b-a}$$

2. Exponential



$$p_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$F_X(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-\lambda x}, & x \geq 0 \end{cases}$$

$$\int_0^x \lambda e^{-\lambda x'}\, dx' = \lambda \frac{e^{-\lambda x'}}{-\lambda} \Big|_0^x$$

$$= -\left[e^{-\lambda x} - 1\right]$$

$$= 1 - e^{-\lambda x}$$

3. Gaussian or Normal

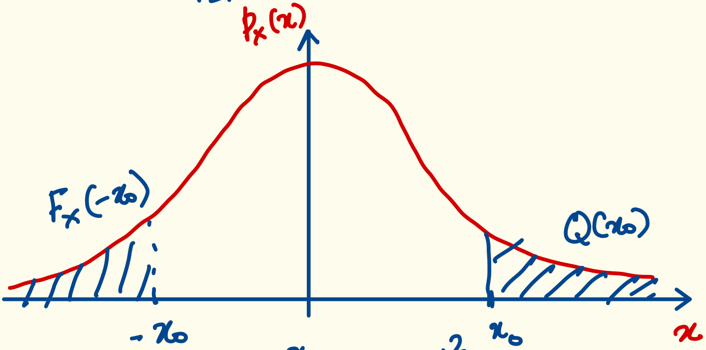
$$p_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}} , \quad -\infty < x < \infty$$

$$X \sim N(\mu, \sigma^2)$$

Standard Normal

$$X \sim N(0, 1)$$

$$p_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} , \quad -\infty < x < \infty$$



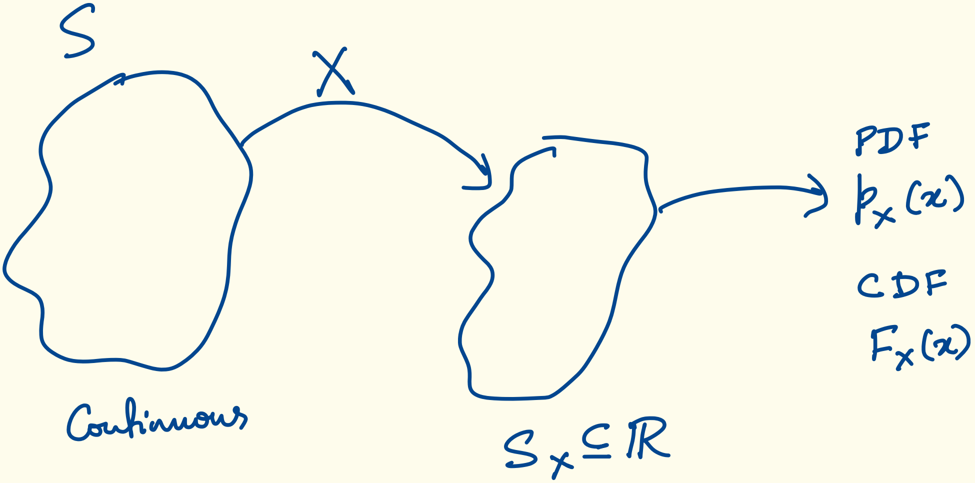
$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, dt$$

$$Q(x) = \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, dt$$

$$F_X(x) = 1 - Q(x)$$

$$F_X(-\infty) = 0 \qquad Q(-\infty) = 1$$

$$F_X(\infty) = 1 \qquad Q(\infty) = 0$$



4. Laplacian

5. Cauchy

6. Rayleigh

HW

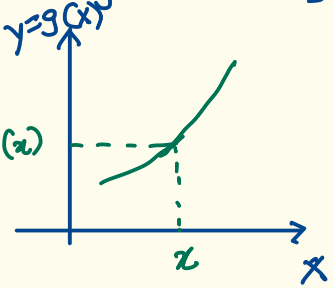
Function of Cont. R.V. X (Transformation)

$$X \quad , \quad \overset{\text{PDF}}{p_X(x)} \quad , \quad \overset{\text{CDF}}{F_X(x)}$$

$$Y = g(X) \quad , \quad p_Y(y) \quad , \quad F_Y(y)$$

Assume is $g(\cdot)$ is one-one (invertible)
g^{-1} exists.

Case a) $Y = g(X)$ is monotonically increasing function.



$$F_Y(y) = P[Y \leq y]$$

$$= P[g(X) \leq y]$$

$$= P[X \leq g^{-1}(y)]$$

$$F_Y(y) = F_X(g^{-1}(y)) \quad \text{———} \quad ①$$

$$p_Y(y) = \frac{d}{dy} F_Y(y)$$

From ①

$$p_Y(y) = \frac{d}{dy} F_X(g^{-1}(y))$$

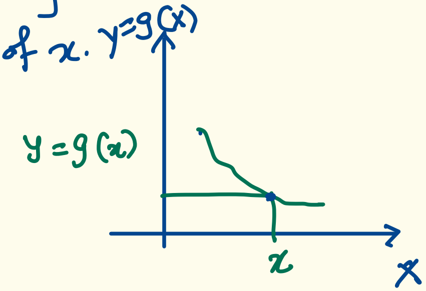
$$= \frac{d}{dy} F_X(x) \Big|_{x=g^{-1}(y)}$$

$$= \frac{d}{dx} F_X(x) \cdot \frac{d}{dy} g^{-1}(y)$$

$$p_Y(y) = p_X(x) \cdot \frac{d}{dy} g^{-1}(y) \longrightarrow ③$$

Case b

$Y = g(X)$ is monotonically decreasing function of x.



$$F_Y(y) = P[Y \leq y]$$

$$= P[g(X) \leq y]$$

$$= P[X > g^{-1}(y)]$$

$$= 1 - P[X \leq g^{-1}(y)]$$

$$F_Y(y) = 1 - F_X(g^{-1}(y)) \longrightarrow ①$$

$$p_y(y) = \frac{d}{dy} F_y(y)$$

$$= \frac{d}{dy}\left(1 - F_x(g^{-1}(y))\right)$$

$$= -\frac{d}{dy} F_x(g^{-1}(y))$$

$$= \frac{d}{dx} F_x(x) \cdot -\frac{d}{dy} g^{-1}(y)$$

$$p_y(y) = p_x(x) \underbrace{\left(-\frac{d}{dy} g^{-1}(y)\right)}_{+ve} \qquad \text{— ④}$$

From ③ and ④

$x = g^{-1}(y)$

$$p_y(y) = p_x(g^{-1}(y)) \left|\frac{d}{dy} g^{-1}(y)\right|$$

eg: $Y = aX + b$, $p_x(x)$

$x = \frac{y-b}{a}$ $\qquad g^{-1}(y) = \frac{y-b}{a}$

$$p_y(y) = p_x\left(\frac{y-b}{a}\right) \left|\frac{d}{dy}\left(\frac{y-b}{a}\right)\right| = \frac{1}{|a|} p_x\left(\frac{y-b}{a}\right)$$

eg: $X \sim N(0,1)$

$$p_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad , \quad -\infty < x < \infty$$

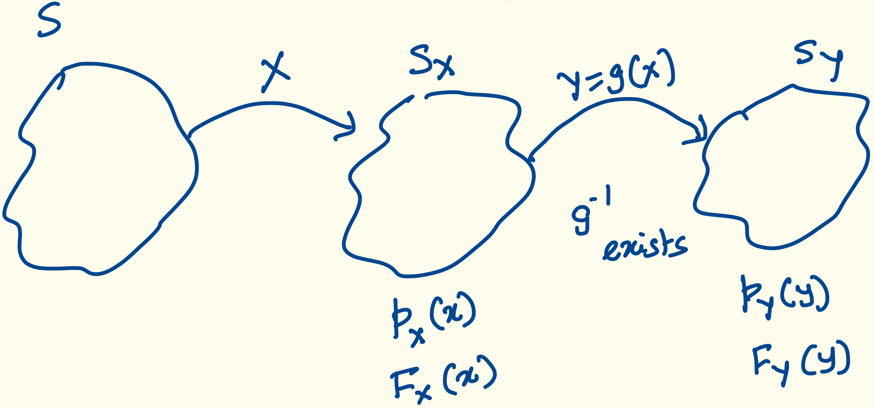
$$Y = \sqrt{\sigma^2}\, X + \mu$$

$$g^{-1}(y) = \frac{y-\mu}{\sigma} \qquad \frac{d}{dy} g^{-1}(y) = \frac{1}{|\sigma|} = \frac{1}{\sigma}$$

$$p_Y(y) = p_X\left(g^{-1}(y)\right) \frac{1}{\sigma}$$

$$p_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}} \quad , \quad -\infty < y < \infty$$

$$Y \sim N(\mu, \sigma^2)$$



$$F_y(y) = F_x\left(g^{-1}(y)\right)$$

$$p_y(y) = p_x\left(g^{-1}(y)\right) \left| \frac{d}{dy} g^{-1}(y) \right|$$

eg.

$$X \sim N(\mu, \sigma^2)$$

$$p_x(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2} \frac{(x-\mu)^2}{\sigma^2}} \qquad -\infty < x < \infty$$

$$Y = e^X \qquad -\infty < X < \infty \qquad 0 < Y < \infty$$

$$X = \ln(Y) \rightarrow g^{-1}(y)$$

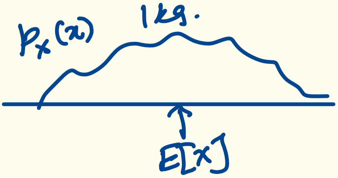
$$p_y(y) = p_x(\ln(y)) \left| \frac{d}{dy} \ln(y) \right|$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(\ln(y)-\mu)^2}{2\sigma^2}} \left| \frac{1}{y} \right|$$

$$p_y(y) = \frac{1}{\sqrt{2\pi}\,\sigma} \frac{1}{|y|} e^{-\frac{(\ln(y)-\mu)^2}{2\sigma^2}} \quad , 0 < y < \infty$$

log-normal pdf

Expectation & Variance



$X \rightarrow$ Cont. R.V.

$$E[X] = \int_{-\infty}^{+\infty} x \, p_X(x) \, dx \quad \text{exists iff} \int_{-\infty}^{+\infty} |x| \, p_X(x) \, dx < \infty$$

$$\text{var}(X) = E\left[(X - E[X])^2\right]$$

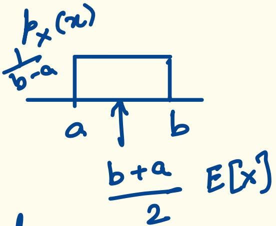
$$= \int_{-\infty}^{+\infty} (x - E[X])^2 \, p_X(x) \, dx \qquad \text{Error}$$

$$Y = g(X)$$

$$E[Y] = E[g(X)] = \int_{-\infty}^{+\infty} g(x) \, p_X(x) \, dx$$

eg: Uniform r.v.

$$p_X(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & o/w \end{cases}$$



$$E[X] = \int_a^b x \frac{1}{b-a} \, dx = \frac{1}{b-a} \int_a^b x \, dx$$

$$= \frac{1}{b-a} \left. \frac{x^2}{2} \right|_a^b = \frac{(b^2 - a^2)}{2(b-a)} = \frac{b+a}{2}$$

$$\mathrm{Var}(x) = \int_a^b (x - E[x])^2 \frac{1}{b-a}\, dx$$

$$= \int_a^b \left(x - \frac{b+a}{2}\right)^2 \frac{1}{b-a}\, dx$$

$u = x - \frac{b+a}{2}$

$x = a$

$u = \frac{a-b}{2}$

$x = b$

$u = \frac{b-a}{2}$

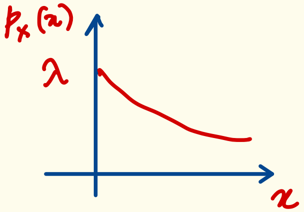
$$= \frac{1}{b-a} \int_{\frac{a-b}{2}}^{\frac{b-a}{2}} u^2\, du$$

$$= \frac{1}{b-a} \frac{u^3}{3} \Bigg|_{\frac{a-b}{2}}^{\frac{b-a}{2}}$$

$$= \frac{1}{3(b-a)} \left[\frac{(b-a)^3}{8} - \frac{(a-b)^3}{8} \right]$$

$$= \frac{1}{24} \frac{1}{b-a} \left[2(b-a)^3 \right]$$

$$\mathrm{Var}(x) = \frac{(b-a)^2}{12}$$



eg: $X \sim \exp(\lambda)$

$$p_x(x) = \begin{cases} 0, & x<0 \\ \lambda e^{-\lambda x}, & x \geq 0 \end{cases}$$

$$E[X] = \int_0^\infty x \lambda e^{-\lambda x} \, dx$$

$$= \int_0^\infty \underset{u}{x} \, d(\underset{v}{-e^{-\lambda x}})$$

$$= \underbrace{-x e^{-\lambda x} \Big|_0^\infty}_{0} - \int_0^\infty (-e^{-\lambda x}) \, dx$$

$$= \int_0^\infty e^{-\lambda x} \, dx$$

$$= \frac{e^{-\lambda x}}{-\lambda} \Big|_0^\infty$$

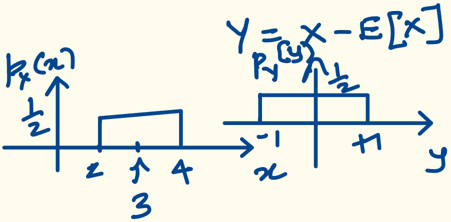
$$= 0 - \left[-\frac{1}{\lambda}\right]$$

$$E[X] = \frac{1}{\lambda}$$

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2$$

Moments

$$\underset{g(x)}{E[X^n]} = \int_{-\infty}^{+\infty} x^n \, p_X(x)\,dx$$



Central Moments

$$\underset{g(x)}{E[(X - E[X])^n]} = \int_{-\infty}^{+\infty} (x - E[X])^n \, p_X(x)\,dx$$

eg: $X \sim \exp(\lambda)$

$$E[X^2] = \int_{0}^{+\infty} x^2 \lambda e^{-\lambda x}\,dx$$

$$= \int_{0}^{+\infty} \underset{u}{x^2} \, d(\underset{v}{-e^{-\lambda x}})$$

$$= \underbrace{-\underset{u}{x^2}\,\underset{v}{e^{-\lambda x}}\Big|_{0}^{\infty}}_{0} - \int_{0}^{\infty} (-e^{-\lambda x})\, d(x^2)$$

$$E[X^2] = \int_0^\infty e^{-\lambda x} \, d(x^2)$$

$$= \int_0^\infty 2x \, e^{-\lambda x} \, dx$$

$$= \frac{2}{\lambda} \underbrace{\int_0^\infty x \, \lambda \, e^{-\lambda x} \, dx}_{E[x]}$$

$$E[X^2] = \frac{2}{\lambda} E[x]$$

$$Var(X) = E[X^2] - (E[x])^2$$

$$= \frac{2}{\lambda} \cdot \frac{1}{\lambda} - \frac{1}{\lambda^2}$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

$$E[X^n] = \int_0^\infty x^n \, \lambda \, e^{-\lambda x} \, dx$$

$$= \int_0^\infty \underset{u}{x^n} \, d(\underset{v}{-e^{-\lambda x}})$$

$$= \underbrace{-x^2 e^{-\lambda x} \Big|_0^\infty}_{0} - \int_0^\infty (-e^{-\lambda x}) \, d(x^n)$$

$$= \int_0^\infty e^{-\lambda x} \, n \, x^{n-1} \, dx$$

$$E[x^n] = \frac{n}{\lambda}\int_0^{\infty} x^{n-1}\lambda e^{-\lambda x}\,dx$$

$$E[x^n] = \frac{n}{\lambda}E[x^{n-1}]$$

$$E[x^n] = \frac{n\cdot(n-1)\cdots\cdots 1}{\lambda^n}$$

$$E[x^n] = \frac{n!}{\lambda^n}$$

Characteristic Function $\Phi_x(\omega)$

$$\Phi_x(\omega) = E[\underbrace{e^{j\omega x}}_{g(x)}]$$

$$\Phi_x(\omega) = \int_{-\infty}^{+\infty} e^{j\omega x}\, p_x(x)\, dx \qquad \overset{\omega \to -\omega}{\text{CTFT}}$$

$$p_x(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Phi_x(\omega)\, e^{-j\omega x}\, d\omega \qquad \overset{\omega \to -\omega}{\text{ICTFT}}$$

$\Phi_X(\omega)$ for $p_X(x)$ is not periodic.

Moments from $\Phi_X(\omega)$

$$\frac{d}{d\omega}\Phi_X(\omega)\bigg|_{\omega=0} = \frac{d}{d\omega}\left(\int_{-\infty}^{+\infty} p_X(x)\, e^{j\omega x}\, dx\right)\bigg|_{\omega=0}$$

$$= j\int_{-\infty}^{+\infty} x\, p_X(x)\, \underbrace{e^{j\omega x}}_{1}\, dx\,\bigg|_{\omega=0}$$

$$= j\, E[X]$$

$$E[X] = \frac{1}{j}\frac{d}{d\omega}\Phi_X(\omega)\bigg|_{\omega=0}$$

$$E[X^n] = \frac{1}{j^n}\frac{d^n}{d\omega^n}\Phi_X(\omega)\bigg|_{\omega=0}$$

eg. $X \sim \exp(\lambda)$

$$\Phi_X(\omega) = \int_0^{\infty} e^{j\omega x}\, \lambda e^{-\lambda x}\, dx$$

$$= \lambda\int_0^{\infty} e^{-(\lambda - j\omega)x}\, dx$$

$$= \lambda \left. \frac{e^{-(\lambda - j\omega)x}}{-(\lambda - j\omega)} \right|_0^\infty$$

$$= \lambda \left[0 - \left(- \frac{1}{\lambda - j\omega} \right) \right]$$

$$\Phi_X(\omega) = \frac{\lambda}{\lambda - j\omega}$$

$$E[X] = \frac{1}{j} \frac{d}{d\omega} \left(\frac{\lambda}{\lambda - j\omega} \right) \Bigg|_{\omega=0}$$

$$= \frac{\lambda}{j} - (\lambda - j\omega)^{-2} (-j) \Big|_{\omega=0}$$

$$= \frac{\lambda}{\lambda^2}$$

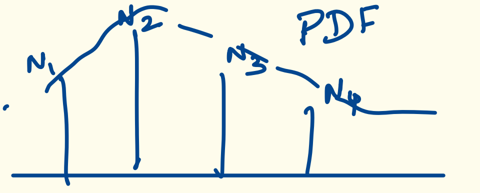
$$E[X] = \frac{1}{\lambda}$$

$$E[X^n] = j \frac{d^n}{d\omega^n} \left(\frac{\lambda}{\lambda - j\omega} \right) \Bigg|_{\omega=0} = \frac{n!}{\lambda^n}$$

$X \rightarrow x_1, x_2, \ldots, x_M$ samples

$\downarrow$

Histogram



$$M = N_1 + N_2 + N_3 + N_4$$

$M \rightarrow \infty$

PMF $\quad \frac{N_1}{M} \quad \frac{N_2}{M} \quad \frac{N_3}{M} \quad \frac{N_4}{M}$

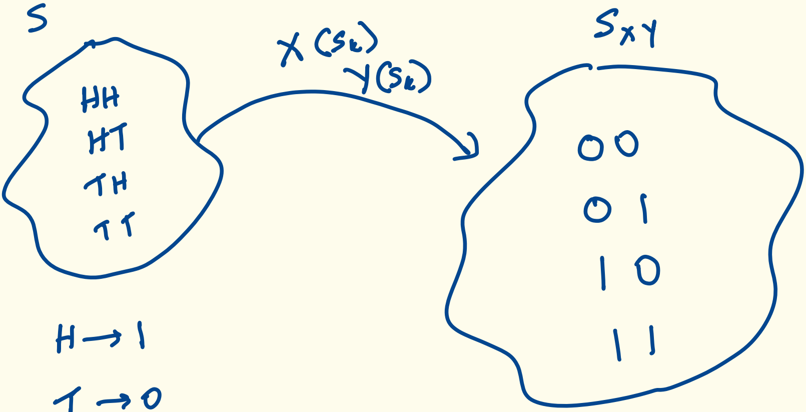
Sample Mean $\widehat{E[X]} = \frac{1}{M} \sum_{i=1}^{M} x_i$

$E[x^2] \qquad (E[x])^2$

Sample Variance $\widehat{\mathrm{Var}(X)} = \frac{1}{M} \sum_{i=1}^{M} x_i^2 - \left(\frac{1}{M} \sum_{i=1}^{M} x_i \right)^2$

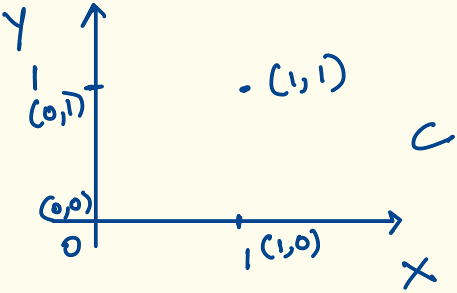
Two Discrete Random Variables

eg. Two coin Toss

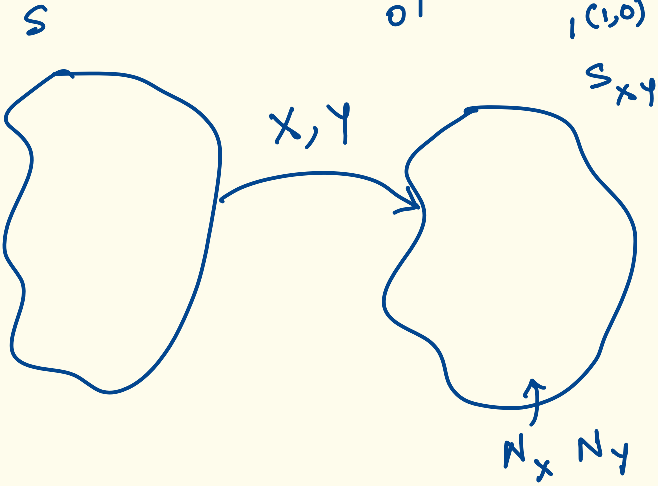


H → 1

T → 0



$\subset \mathbb{Z}^2$



$$X \in \{x_1, x_2, \ldots, x_{N_x}\}$$

$$Y \in \{y_1, y_2, \ldots, y_{N_y}\}$$

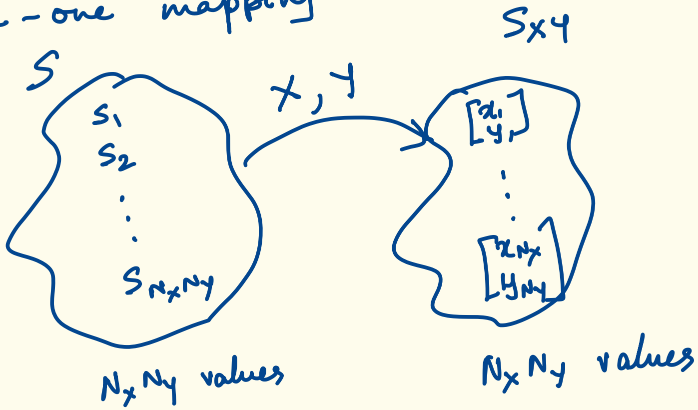
$$\begin{bmatrix}X\\Y\end{bmatrix} \in \underbrace{\left\{ \begin{bmatrix}x_1\\y_1\end{bmatrix}, \begin{bmatrix}x_1\\y_2\end{bmatrix}, \ldots, \begin{bmatrix}x_{N_x}\\y_{N_y}\end{bmatrix} \right\}}_{N_x N_y}$$

$X \longrightarrow N_x$ values

$Y \longrightarrow N_y$ values

$\begin{bmatrix}X\\Y\end{bmatrix} \longrightarrow N_x N_y$ values

One-one mapping

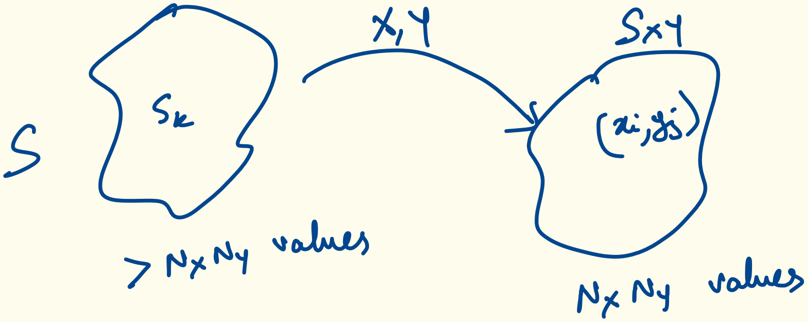


$$\begin{bmatrix} x_i \\ y_j \end{bmatrix} = \begin{bmatrix} X(s_k) \\ Y(s_k) \end{bmatrix} \qquad \begin{matrix} i = 1, 2, \ldots, N_X \\ j = 1, 2, \ldots, N_Y \end{matrix}$$

$$P[X = x_i, Y = y_j] = p_{XY}[x_i, y_j]$$

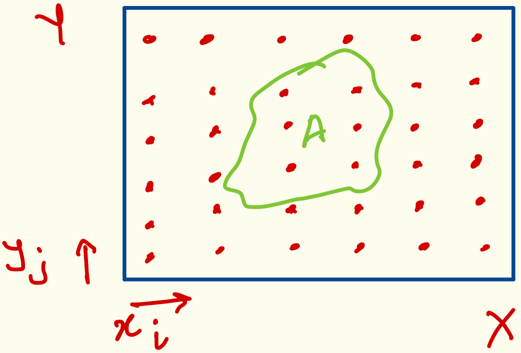
$$p_{XY}[x_i, y_j] = P[\{s_k\}] \qquad k = 1, 2, \ldots, N_X N_Y$$

Many-one mapping



$$p_{XY}[x_i, y_j] = \sum_{\substack{\{s_k :\, x_i = X(s_k) \\ y_j = Y(s_k)\}}} P[\{s_k\}] \qquad \begin{matrix} i = 1, 2, \ldots, N_X \\ j = 1, 2, \ldots, N_Y \\ k = 1, 2, \ldots, M \end{matrix}$$

$p_{XY}[x_i, y_j]$ → Joint Probability Mass Function (PMF)



$N_x = 6$

$N_y = 6$

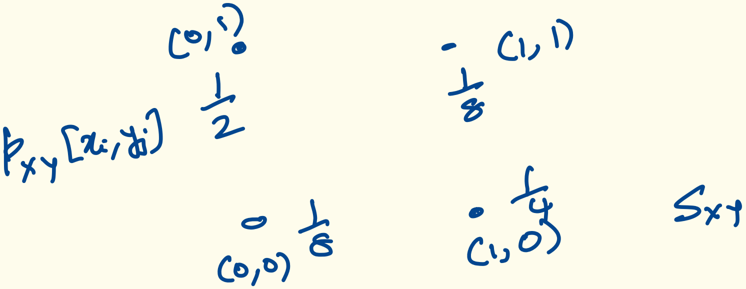
$N_x N_y = 36$ values

Joint PMF

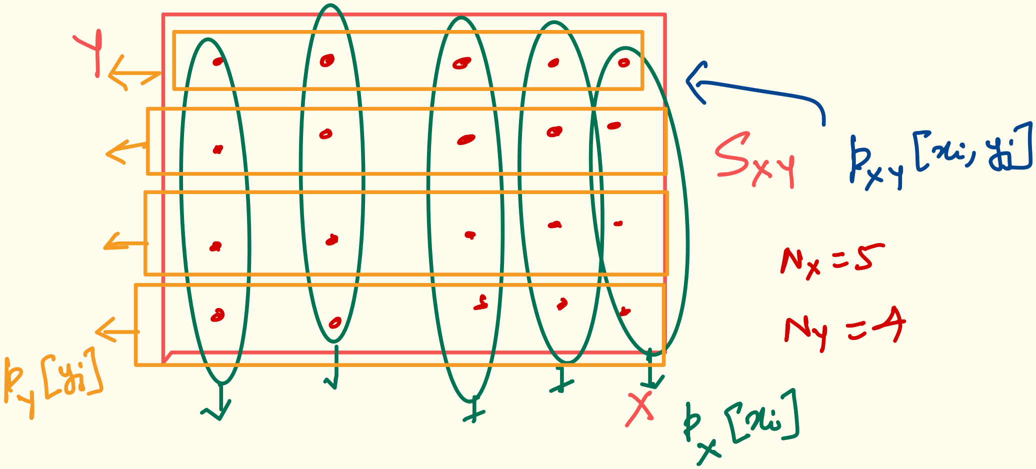
1. $0 \leq p_{XY}[x_i, y_j] \leq 1$

2. $$\sum_{i=1}^{N_X} \sum_{j=1}^{N_Y} p_{XY}[x_i, y_j] = 1$$

eg. Two coin toss



$$P[A] = \sum_{\{(x_i,y_j):(x_i,y_j)\in A\}} \sum p_{XY}[x_i,y_j]$$



$p_{XY}[x_i,y_j]$

Marginal PMF

$$p_X[x_i] = \sum_{j=1}^{N_Y} p_{XY}[x_i,y_j]$$

$$p_Y[y_j] = \sum_{i=1}^{N_X} p_{XY}[x_i,y_j]$$

eg: Two coin toss

	$X \rightarrow$ T 0	H 1	$p_Y[y_i]$
Y ↓ T 0	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{5}{8}$
H 1	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$
$p_X[x_i]$	$\frac{5}{8}$	$\frac{3}{8}$	

$F_{XY}(0.5, 0.5) = \frac{1}{2}$

In general

$$p_{XY}[x_i, y_i] \longrightarrow p_X[x_i], \; p_Y[y_i]$$

$$p_X[x_i], \; p_Y[y_i] \underset{\text{always}}{\cancel{\longrightarrow}} p_{XY}[x_i, y_i]$$

Joint Cumulative Distribution Function (CDF)

$$F_{XY}(x, y) = P[X \le x, Y \le y] \qquad \mathbb{R}^2$$

$$= \sum_{\left\{(x_i, y_i) : \begin{array}{l} x_i \le x \\ y_i \le y \end{array}\right\}} \sum p_{XY}[x_i, y_i]$$

Properties

1. $0 \leq F_{XY}(x,y) \leq 1$

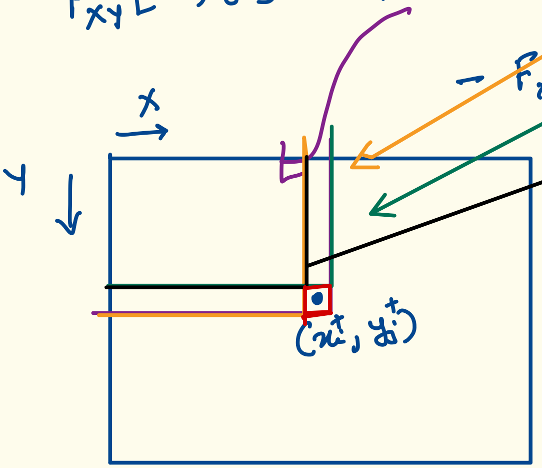
2. $F_{XY}(-\infty, -\infty) = 0$

   $F_{XY}(\infty, \infty) = 1$

3. $F_{XY}(x,y)$ is a monotonically non-decreasing function of x and y.

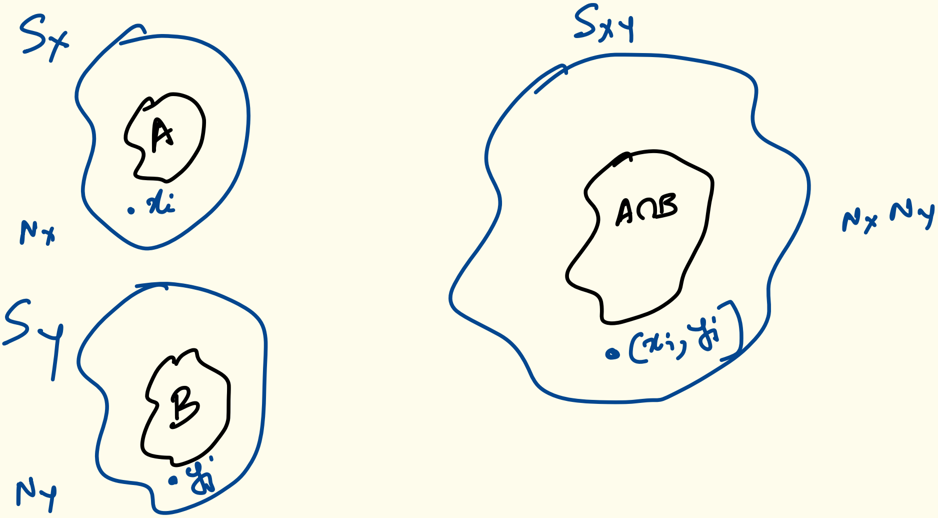
4. Joint PMF from Joint CDF

$$p_{XY}[x_i, y_j] = F_{XY}(x_i^+, y_j^+) - F_{XY}(x_i^-, y_j^+) - F_{XY}(x_i^+, y_j^-) + F_{XY}(x_i^-, y_j^-)$$



Two independent Random Variables

$X, Y \rightarrow$ Discrete Random Variables



$$P[X \in A, Y \in B] = P[X \in A] . P[Y \in B]$$

Let $A = x_i$, $B = y_j$

$$P[X = x_i, Y = y_j] = P[X = x_i] \, P[Y = y_j]$$

$$\underbrace{p_{XY}[x_i, y_j]}_{\text{Joint}} = \underbrace{p_X[x_i]}_{\text{Marginal}} \cdot \underbrace{p_Y[y_j]}_{\text{Marginal}}$$

eg. Coin Toss (Independent)

$$\begin{array}{c|c|cc}
 & & X \rightarrow \frac{1}{2} & \frac{1}{2} \\
 & & 0 & 1 \\
\hline
Y \downarrow \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\
\frac{1}{2} & 1 & \frac{1}{4} & \frac{1}{4}
\end{array}$$

$$p_X[x_i] = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\begin{array}{ccc} x_i & 0 & 1 \end{array}$$

$$p_Y[y_j] = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\begin{array}{ccc} y_j & 0 & 1 \end{array}$$

eg. Dependent Coin Toss

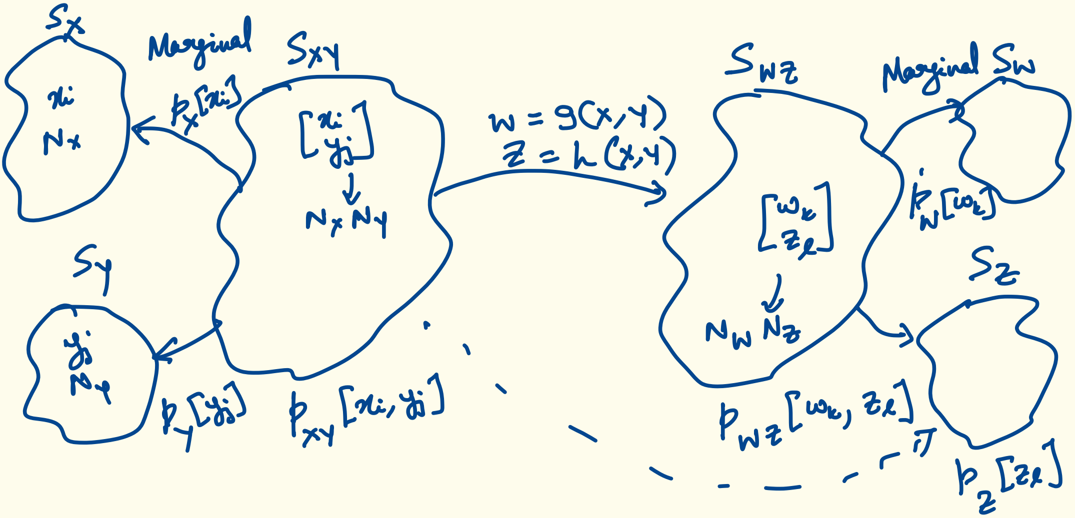
$$\begin{array}{c|c|cc}
 & X \rightarrow p_X[x_i] & 5/8 & 3/8 \\
Y & p_Y[y_j] & 0 & 1 \\
\hline
\downarrow \frac{5}{8} & 0 & \frac{1}{2} & \frac{1}{8} \\
\frac{3}{8} & 1 & \frac{1}{8} & \frac{1}{4}
\end{array}$$

$X, Y \rightarrow$ Not Independent

Functions of Two Discrete Random Variables

(Transformation)

X, Y → Discrete R.V.s



$$p_{WZ}[w_k, z_l] = \sum\sum_{\substack{\{(x_i, y_j) : w_k = g(x_i, y_j), \\ z_l = h(x_i, y_j)\}}} p_{XY}[x_i, y_j]$$

$$p_W[w_k] = \sum_{z_l} p_{WZ}[w_k, z_l]$$

$$p_Z[z_l] = \sum_{w_k} p_{WZ}[w_k, z_l]$$

$$Z = h(X, Y)$$

Method 1 1D

1. Find $\overset{\text{2D}}{p_{W,Z}[w_k, z_l]}$, by setting $\overset{\text{Dummy}}{W = X}$

2. Marginalize $p_Z[z_l] = \sum_{w_k} p_{W,Z}[w_k, z_l]$

Method 2

$$p_Z[z_l] = \sum \sum_{\{(x_i, y_j):\, z_l = h(x_i, y_j)\}} p_{XY}[x_i, y_j]$$

eg.

$$p_{XY}[x_i, y_j] = \begin{cases} 3/8, & x_i = 0,\ y_j = 0 \\ 1/8, & x_i = 1,\ y_j = 0 \\ 1/8, & x_i = 0,\ y_j = 1 \\ 3/8, & x_i = 1,\ y_j = 1 \end{cases}$$

$$Z = X^2 + Y^2$$

$$p_Z[z_l] = \begin{cases} 3/8, & z_l = 0 \\ 2/8, & z_l = 1 \\ 3/8, & z_l = 2 \end{cases}$$

X, Y → Independent Discrete R.Vs.

$$p_{XY}[x_i, y_j] = p_X[x_i]\; p_Y[y_j]$$

$$p_{XY}[i, j] \equiv p_{XY}[x_i, y_j]$$

$$p_{WZ}[k, l] \equiv p_{WZ}[w_k, z_l]$$

$$\underset{l}{Z} = \underset{i}{X} + \underset{j}{Y}$$

$$p_Z[z_l] = ?$$

$$\underset{k}{W} = \underset{i}{X}$$

Step 1 $\quad p_{WZ}[w_k, z_l] \equiv p_{WZ}[k, l]$

Step 2 $\quad p_Z[z_l] = \sum_{w_k} p_{WZ}[w_k, z_l]$

Step 1

$$p_{WZ}[k, l] = \sum \sum_{\{(i,j)\,:\; k = i,\; l = \underset{k}{i} + j\}} p_{XY}[\overset{k}{i}, \overset{l-k}{j}]$$

$$p_{WZ}[k,l] = p_{XY}[k, l-k]$$

$X, Y \rightarrow$ Independent

$$\Rightarrow p_{WZ}[k,l] = p_X[k]\, p_Y[l-k] \text{ ——— } ①$$

Step 2

$$p_Z[l] = \sum_k p_{WZ}[k,l] \text{ ——— } ②$$

① in ②

$$p_Z[l] = \sum_k p_X[k]\, p_Y[l-k] \text{ ——— } ③$$

1D Discrete Convolution

$$p_Z[l] = p_X[l] * p_Y[l]$$

1D Discrete Convolution

$$\begin{array}{lccccc} x[l] & 3 & 4 & 1 & 2 & \quad 4 \\ & \uparrow & & & & \\ & 0 & 1 & 2 & 3 & \\ h[l] & 1 & 2 & & & \\ & \uparrow & \uparrow & & & \\ & 0 & 1 & & & \quad 2 \end{array}$$

$$y[l] = x[l] * h[l]$$

$$= \sum_{k} x[k]\, h[l-k]$$

$n_x + n_h - 1$

$4 + 2 - 1$

$$= \langle x[k], h[l-k] \rangle$$

$= 5$ inner products

$$\begin{array}{lcccc} x[k] & 3 & 4 & 1 & 2 \\ & \uparrow & & & \\ & 0 & & & \end{array}$$

$$\begin{array}{lcc} h[k] & 1 & 2 \\ & \uparrow & \\ & 0 & \end{array}$$

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 4 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{array}{lcc} h[-k] & 2 & 1 \\ & & \uparrow \\ & & 0 \end{array}$$

$$y[0] = \sum_{k} x[k]\, h[-k] = 3$$

$$\downarrow$$

$$\begin{array}{lcc} h[1-k] & 2 & 1 \\ & \uparrow & \\ & 0 & \end{array}$$

$$\begin{bmatrix} 0 & 2 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 4 \\ 1 \\ 2 \end{bmatrix}$$

$$y[1] = \sum_{k} x[k]\, h[1-k] = 3 \times 2 + 4 \times 1 = 10$$

$$h[2-k] \quad \underset{\overset{\uparrow}{0}}{0} \quad 2 \quad 1$$

$$x[k] \quad \underset{\overset{\uparrow}{0}}{3} \quad 4 \quad 1 \quad 2$$

$$\begin{bmatrix} 0 & 0 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 4 \\ 1 \\ 2 \end{bmatrix}$$

$$y[2] = 9$$

$$h[3-k] \quad \underset{\overset{\uparrow}{0}}{0} \quad 0 \quad 2 \quad 1$$

$$\begin{bmatrix} 0 & 0 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 4 \\ 1 \\ 2 \end{bmatrix}$$

$$y[3] = \sum_k x[k]\, h[3-k] = 2 \cdot 1 + 2 \cdot 1 = 4$$

$$h[4-k] \quad \underset{\overset{\uparrow}{0}}{0} \quad 0 \quad 0 \quad 2 \quad 1$$

$$y[4] = 4$$

$$y[l] = \underset{\overset{\uparrow}{0}}{3} \quad 10 \quad 9 \quad 4 \quad 4 \quad \checkmark$$

$$n_x = 4$$

$$n_h = 2$$

$$n_y = 4 + 2 - 1 = 5$$

Characteristic Function

$$\Phi_z(\omega) = \sum_{l} p_z[l]\, e^{j\omega l} \quad \text{DTFT}$$

$$= \sum_{l} \sum_{k} p_x[k]\, p_y[l-k]\, e^{j\omega l} \quad \text{from } ③$$

$$l-k = q \Rightarrow l = q+k$$

$$\Phi_z(\omega) = \sum_{q} \sum_{k} p_x[k]\, p_y[q]\, e^{j\omega(q+k)}$$

$$= \sum_{k} p_x[k]\, e^{j\omega k} \sum_{q} p_y[q]\, e^{j\omega q}$$

$$\Phi_z(\omega) = \Phi_x(\omega)\, \Phi_y(\omega)$$

$$p_z[l] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_z(\omega)\, e^{-j\omega l}\, d\omega$$

eg.

$$\left.\begin{array}{l} X \sim \text{Pois}(\lambda_x) \\ Y \sim \text{Pois}(\lambda_y) \end{array}\right] \rightarrow \text{Independent}$$

$$Z = X + Y$$

$$p_x[i] = \frac{e^{-\lambda_x}(\lambda_x)^i}{i!}, \quad i = 0, 1, 2, \ldots$$

$$p_y[j] = \frac{e^{-\lambda_y}(\lambda_y)^j}{j!}, \quad j = 0, 1, 2, \ldots$$

$$\Phi_z(\omega) = \Phi_x(\omega)\,\Phi_y(\omega)$$

$$\Phi_x(\omega) = \sum_{i=0}^{\infty} \frac{e^{-\lambda_x}(\lambda_x)^i}{i!}\, e^{j\omega i}$$

$$= \sum_{i=0}^{\infty} \frac{e^{-\lambda_x}(\lambda_x e^{j\omega})^i}{i!}\, e^{\lambda_x e^{j\omega}} \cdot e^{-\lambda_x e^{j\omega}}$$

$$= \sum_{i=0}^{\infty} \frac{e^{-\lambda_x e^{j\omega}}(\lambda_x e^{j\omega})^i}{i!}\, e^{\lambda_x(e^{j\omega}-1)}$$

$$\Phi_x(\omega) = e^{\lambda_x(e^{j\omega}-1)}$$

$$\Phi_y(\omega) = e^{\lambda_y(e^{j\omega}-1)}$$

$$\Phi_z(\omega) = \Phi_x(\omega)\,\Phi_y(\omega)$$

$$= e^{\lambda_x(e^{j\omega}-1)} \cdot e^{\lambda_y(e^{j\omega}-1)}$$

$$= e^{\lambda_x(e^{j\omega}-1) + \lambda_y(e^{j\omega}-1)}$$

$$\Phi_z(\omega) = e^{(\lambda_x+\lambda_y)(e^{j\omega}-1)}$$

$$p_z[l] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_z(\omega)\, e^{-j\omega l}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(\lambda_x+\lambda_y)(e^{j\omega}-1)}\, e^{-j\omega l}\, d\omega$$

$$\Rightarrow \quad Z \sim \text{Pois}(\lambda_x + \lambda_y)$$

$$p_Z[l] = \frac{e^{-(\lambda_x+\lambda_y)} (\lambda_x+\lambda_y)^l}{l!}, \quad l = 0, 1, 2, \ldots$$

Joint Expectation

$X, Y \rightarrow$ Discrete RVs

$$E_{XY}[XY] = \sum_{x_i} \sum_{y_j} x_i y_j \, p_{XY}[x_i, y_j]$$

$g(X, Y)$

$$E_{XY}[g(X, Y)] = \sum_{x_i} \sum_{y_j} g(x_i, y_j) \, p_{XY}[x_i, y_j]$$

$$g(X, Y) = X + Y$$

$$E_{XY}[X+Y] = \sum_{x_i} \sum_{y_j} (x_i + y_j) \, p_{XY}[x_i, y_j]$$

$$E_{xy}[x+y] = \sum_{x_i}\sum_{y_j} x_i\, p_{xy}[x_i, y_j] + \sum_{x_i}\sum_{y_j} y_j\, p_{xy}[x_i, y_j]$$

$$= \sum_{x_i} x_i \sum_{y_j} p_{xy}[x_i, y_j] + \sum_{y_j} y_j \sum_{x_i} p_{xy}[x_i, y_j]$$

$$= \sum_{x_i} x_i\, p_x[x_i] + \sum_{y_j} y_j\, p_y[y_j]$$

$$E_{xy}[x+y] = E_x[x] + E_y[y]$$

$$E_{xy}[g(x) + h(y)] = E_x[g(x)] + E_y[h(y)]$$

$$E_{xy}[g(x,y)] = \sum_{x_i}\sum_{y_j} g(x_i, y_j)\, p_{xy}[x_i, y_j]$$

$$E_{xy}[g(x)\,h(y)] = \sum_{x_i}\sum_{y_j} g(x_i)\, h(y_j)\, p_{xy}[x_i, y_j]$$

Let x, y → Indep.

$$p_{xy}[x_i, y_j] = p_x[x_i]\, p_y[y_j]$$

$$E_{xy}[g(x)h(y)] = \sum_{x_i}\sum_{y_i} g(x_i)\,h(y_i)\,p_x[x_i]\,p_y[y_i]$$

$$= \sum_{x_i} g(x_i)\,p_x[x_i] \sum_{y_i} h(y_i)\,p_y[y_i]$$

$$= E_x[g(x)]\;E_y[h(y)]$$

$$E_{xy}[xy] = E_x[x]\;E_y[y]$$

$$E_{xy}[g(x)] = E_x[g(x)]$$

$$E_{xy}[h(y)] = E_y[h(y)]$$

$$z = x+y$$

$$E_z[(z-E_z[z])^2]$$

$$\mathrm{Var}(z) = \mathrm{Var}(x+y)$$

$$= E_{xy}\left[\left((x+y) - E_{xy}(x+y)\right)^2\right]$$

$$\mathrm{var}(x+y) = E_{xy}\left[\left((x+y) - (E_x[x] + E_y[y])\right)^2\right]$$

$$= E_{xy}\left[\left((x - E_x[x]) + (y - E_y[y])\right)^2\right]$$

$$= E_{xy}\left[(x - E_x[x])^2 + (y - E_y[y])^2 + 2(x - E_x[x])(y - E_y[y])\right]$$

$$= E_{xy}\left[(x - E_x[x])^2\right] + E_{xy}\left[(y - E_y[y])^2\right] + 2\,E_{xy}\left[(x - E_x[x])(y - E_y[y])\right]$$

$$\mathrm{Var}(x+y) = E_x\left[(x - E_x[x])^2\right] + E_y\left[(y - E_y[y])^2\right] + 2\,\mathrm{Cov}(x,y)$$

$$\mathrm{var}(x+y) = \mathrm{var}(x) + \mathrm{var}(y) + 2\,\mathrm{Cov}(x,y)$$

$$\mathrm{var}(x+y) > \mathrm{var}(x) + \mathrm{var}(y) \quad \text{if } \mathrm{Cov}(x,y) > 0$$

$$\mathrm{var}(x+y) < \mathrm{var}(x) + \mathrm{var}(y) \quad \text{if } \mathrm{Cov}(x,y) < 0$$

$$\text{var}(x+y) = \text{var}(x) + \text{var}(y) \quad \text{if } \text{Cov}(x,y)=0$$

$x, y \rightarrow$ Uncorrelated

iff $\text{Cov}(x,y) = 0$

$$\text{Cov}(x,y) = E_{xy}\left[(x - E_x[x])(y - E_y[y])\right]$$

1st order Joint Central Moment

$$\text{Cov}(x,y) = E_{xy}\left[xy + E_x[x]\,E_y[y] - x\,E_y[y] - y\,E_x[x]\right]$$

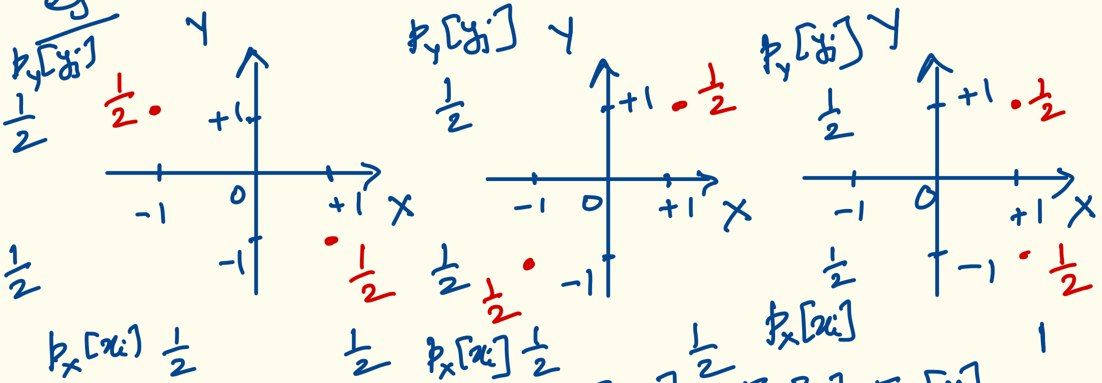
$$= E_{xy}[xy] + E_x[x]\,E_y[y] - E_y[y]\,E_x[x] - E_x[x]\,E_y[y]$$

$$\text{Cov}(x,y) = E_{xy}[xy] - E_x[x]\,E_y[y]$$

If x, y independent, $E_{xy}[xy] = E_x[x]\,E_y[y]$

$$\text{Cov}(x,y) = 0$$

eg.



$$\mathrm{Cov}(X,Y) = E_{XY}[XY] - E_X[X]\,E_Y[Y]$$

$E_X[X] = 0$ $\quad$ $E_X[X] = 0$ $\quad$ $E_X[X] = 1$

$E_Y[Y] = 0$ $\quad$ $E_Y[Y] = 0$ $\quad$ $E_Y[Y] = 0$

$$\mathrm{Cov}(X,Y) = E_{XY}[XY]$$

$$= \sum_{x_i}\sum_{y_i} x_i\, y_i\, p_{XY}[x_i, y_i]$$

$$\mathrm{Cov}(X,Y) = (-1)(+1)\frac{1}{2} + (+1)(-1)\frac{1}{2}$$

$$= -\frac{1}{2} - \frac{1}{2}$$

$$= -1$$

-ve Corr.

$$\mathrm{Cov}(X,Y) = +1$$

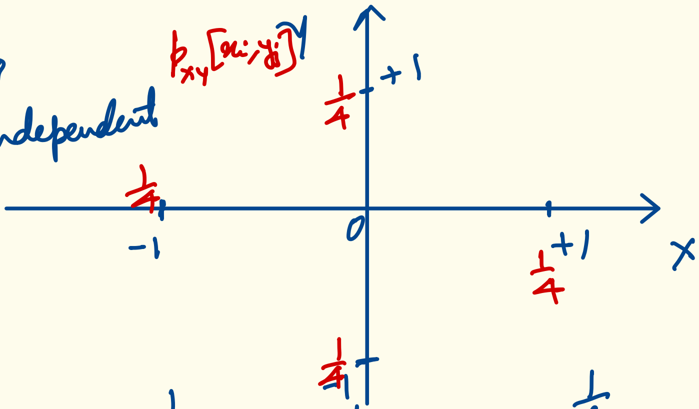
+ve Corr.

$$\mathrm{Cov}(X,Y) = \frac{1}{2} - \frac{1}{2} = 0$$

X, Y → Uncorrelated

X, Y →

Not Independent



$p_Y[y_i]$

$\frac{1}{4}$

$\frac{1}{2}$

$\frac{1}{4}$

$p_X[x_i]$ $\quad \frac{1}{4} \quad \frac{1}{2} \quad \frac{1}{4}$

$$p_{XY}[x_i, y_i] = \begin{cases} \frac{1}{4}, & x_i = 0, \; y_i = 1 \\ \frac{1}{4}, & x_i = 1, \; y_i = 0 \\ \frac{1}{4}, & x_i = -1, \; y_i = 0 \\ \frac{1}{4}, & x_i = 0, \; y_i = -1 \end{cases}$$

$E_X[X] = 0 \qquad E_Y[Y] = 0$

$$\mathrm{Cov}(X, Y) = \sum_{x_i} \sum_{y_i} x_i y_i \, p_{XY}[x_i, y_i]$$

$$= 0$$

X, Y → Uncorrelated

$X, Y \rightarrow$ Two Discrete R.V.s

$X \rightarrow$ known/observed

$\hat{Y} \rightarrow$ Prediction of Y

$$\hat{Y} = aX + b$$

$E\left[(Y - \hat{Y})^2\right]$ needs to be minimized.

Minimum Mean Squared Error (MMSE)

$$\begin{bmatrix} a_{opt} \\ b_{opt} \end{bmatrix} = \arg\min_{a, b} E_{XY}\left[(Y - aX - b)^2\right]$$

$$E_{XY}\left[((Y - aX) - b)^2\right]$$

$$= E_{XY}\left[(Y - aX)^2 + b^2 - 2b(Y - aX)\right]$$

$$= E_{XY}\left[Y^2 + a^2X^2 - 2aXY + b^2 - 2bY + 2abX\right]$$

$$= E_Y[Y^2] + a^2 E_X[X^2] - 2a\, E_{XY}[XY] + b^2$$
$$- 2b\, E_Y[Y] + 2ab\, E_X[X]$$

$$\frac{\partial}{\partial a_{opt}}\left(\quad\right) = 0$$

$$2a_{opt}\,E_x[x^2] - 2\,E_{xy}[xy] + 2\,b_{opt}\,E_x[x] = 0$$

$$a_{opt}\,E_x[x^2] + b_{opt}\,E_x[x] = E_{xy}[xy] \quad\text{———— ①}$$

$$\frac{\partial}{\partial b_{opt}}\left(\quad\right) = 0$$

$$2\,b_{opt} - 2\,E_y[y] + 2a_{opt}\,E_x[x] = 0$$

$$a_{opt}\,E_x[x] + b_{opt} = E_y[y] \quad\text{—— ②}$$

$$\text{②} \Rightarrow \quad b_{opt} = E_y[y] - a_{opt}\,E_x[x] \quad\text{— ③}$$

③ in ①

$$a_{opt}\,E_x[x^2] + \left(E_y[y] - a_{opt}\,E_x[x]\right)E_x[x] = E_{xy}[xy]$$

$$a_{opt}\left(E_x[x^2]-(E_x[x])^2\right) = E_{xy}[xy] - E_x[x]\,E_y[y]$$

$$a_{opt}\ \mathrm{var}(x) = \mathrm{Cov}(x,y)$$

$$a_{opt} = \frac{\mathrm{Cov}(x,y)}{\mathrm{var}(x)} \quad \text{———} ④$$

④ in ③

$$b_{opt} = E_y[y] - \frac{\mathrm{Cov}(x,y)}{\mathrm{var}(x)} E_x[x] \quad \text{——} ⑤$$

$$\hat{y} = a_{opt} X + b_{opt}$$

$$\hat{y} = \frac{\mathrm{Cov}(x,y)}{\mathrm{var}(x)} X + E_y[y] - \frac{\mathrm{Cov}(x,y)}{\mathrm{var}(x)} E_x[x]$$

$$\hat{y} = \left(X - E_x[x]\right)\frac{\mathrm{Cov}(x,y)}{\mathrm{var}(x)} + E_y[y] \quad \text{——} ⑥$$

If x, y → Uncorrelated $\Rightarrow \mathrm{Cov}(x,y)=0$

$$\hat{y} = E_y[y]$$

⑥ $\Rightarrow$

$$\hat{Y} - E_y[Y] = \text{Cov}(X,Y)\,\frac{(X - E_x[X])}{\text{var}(X)}$$

$\div \sqrt{\text{var}(Y)}$

$$\frac{\hat{Y} - E_y[Y]}{\sqrt{\text{var}(Y)}} = \frac{\text{Cov}(X,Y)}{\sqrt{\text{var}(X)}\,\sqrt{\text{var}(Y)}}\,\frac{(X - E_x[X])}{\sqrt{\text{var}(X)}}$$

$$\left.\begin{aligned} \hat{Y}_s &= \frac{\hat{Y} - E_y[Y]}{\sqrt{\text{var}(Y)}} \\ X_s &= \frac{X - E_x[X]}{\sqrt{\text{var}(X)}} \end{aligned}\right] \rightarrow \begin{array}{c}\text{Normalized} \\ \hat{Y}, X\end{array}$$

$$\hat{Y}_s = \frac{\text{Cov}(X,Y)}{\sqrt{\text{var}(X)}\,\sqrt{\text{var}(Y)}}\;X_s$$

$$\hat{Y}_s = \rho_{xy}\,X_s$$

$$\rho_{xy} = \frac{\text{Cov}(X,Y)}{\sqrt{\text{var}(X)}\,\sqrt{\text{var}(Y)}} \rightarrow \text{Correlation Coefficient}$$

Cauchy Schwarz Inequality $v, w \to 2$ r.vs.

$$|E_{vw}[vw]| \leq \sqrt{E_v[v^2]}\,\sqrt{E_w[w^2]}$$

$$|\vec{a}\cdot\vec{b}| \leq \|\vec{a}\|_2\,\|\vec{b}\|_2$$

$$\rho_{xy} = \frac{E_{xy}\left[(x-E_x[x])(y-E_y[y])\right]}{\sqrt{E_x[(x-E_x[x])^2]}\,\sqrt{E_y[(y-E_y[y])^2]}}$$

$v = x - E_x[x]$, $w = y - E_y[y]$

$$|\rho_{xy}|\rho_{vw}| = \frac{|E_{vw}[vw]|}{\sqrt{E_v[v^2]}\,\sqrt{E_w[w^2]}} \leq 1$$

$$|\rho_{xy}| \leq 1$$

$$-1 \leq \rho_{xy} \leq 1$$

Joint Characteristic Function

$$\Phi_{XY}(\omega_X, \omega_Y) = E_{XY}\left[e^{j(\omega_X X + \omega_Y Y)}\right]$$

$$= \sum_k \sum_l p_{XY}[k,l] \, e^{j(\omega_X k + \omega_Y l)}$$

2D DTFT

$$p_{XY}[k,l] = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \Phi_{XY}(\omega_X, \omega_Y) \, e^{-j(\omega_X k + \omega_Y l)} \, d\omega_X \, d\omega_Y$$

Moments

$$E_{XY}[X^m Y^n] = \frac{1}{j^{m+n}} \frac{\partial^{m+n}}{\partial \omega_X^m \, \partial \omega_Y^n} \Phi_{XY}(\omega_X, \omega_Y) \Bigg|_{\substack{\omega_X = 0 \\ \omega_Y = 0}}$$

X, Y independent $p_{XY}[k,l] = p_X[k] \, p_Y[l]$

$$\Phi_{XY}(\omega_X, \omega_Y) = \sum_k p_X[k] \, e^{j\omega_X k} \sum_l p_Y[l] \, e^{j\omega_Y l}$$

$$= \Phi_X(\omega_X) \, \Phi_Y(\omega_Y)$$

Conditional Discrete Random Variables

$X, Y \rightarrow$ 2 Discrete R.Vs.

Joint PMF $\quad p_{XY}[x_i, y_i]$

Marginal PMFs

$$p_X[x_i] = \sum_{y_i} p_{XY}[x_i, y_i]$$

$$p_Y[y_i] = \sum_{x_i} p_{XY}[x_i, y_i]$$

$$p_{XY}[x_i, y_i] = P[X = x_i, Y = y_i]$$

$$p_X[x_i] = P[X = x_i]$$

$$p_Y[y_i] = P[Y = y_i]$$

$$p_{X/Y}[x_i / y_i] = P\left[X = x_i / Y = y_i\right]$$

$$= \frac{P[X = x_i, Y = y_i]}{P[Y = y_i]}$$

$$p_{X/Y}[x_i/y_j] = \frac{p_{XY}[x_i, y_j]}{p_Y[y_j]}$$

$$p_{Y/X}[y_j/x_i] = \frac{p_{XY}[x_i, y_j]}{p_X[x_i]}$$

$p_{X/Y}$, $p_{Y/X}$ → Conditional PMFs

eg. X → Coin Toss $\{\overset{T}{0}, \overset{H}{1}\}$

Y → Die Toss $\{1, 2, 3, 4, 5, 6\}$

$X \rightarrow$	x_1 0	x_2 1	$p_Y[y_j]$
$Y \downarrow$	$p_{X,Y}$		
y_1 1	0.1	0.05	0.15
y_2 2	0.1	0.05	0.15
y_3 3	0.05	0.05	0.1
y_4 4	0.05	0.05	0.1
y_5 5	0.1	0.1	0.2
y_6 6	0.1	0.2	0.3
$p_X[x_i]$	0.5	0.5	

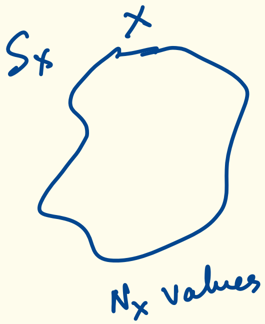
$p_{Y/X}[y_i/x_i]$

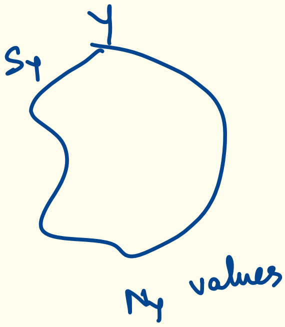
1. $p_{Y/X}[y_i | x_1=0] = \dfrac{p_{XY}[0, y_i]}{p_X[0]}$

$$p_{Y/X}[y_i/0] = \begin{cases} 0.2 & y_i = 1 \\ 0.2 & y_i = 2 \\ 0.1 & y_i = 3 \\ 0.1 & y_i = 4 \\ 0.2 & y_i = 5 \\ 0.2 & y_i = 6 \end{cases}$$

2. $p_{Y/X}[y_i | x_2=1]$

$$p_{Y/X}[y_i | 1] = \begin{cases} 0.1 & y_i = 1 \\ 0.1 & y_i = 2 \\ 0.1 & y_i = 3 \\ 0.1 & y_i = 4 \\ 0.2 & y_i = 5 \\ 0.4 & y_i = 6 \end{cases}$$





$p_{Y/X}$ → N_X PMFs each of length N_Y

$p_{X/Y}$ → N_Y PMFs each of length N_X

Properties

1. $0 \le p_{X/Y}[x_i/y_j] \le 1$

   $0 \le p_{Y/X}[y_j/x_i] \le 1$

2. $$\sum_{x_i} p_{X/Y}[x_i/y_j] = 1$$

   $$\sum_{y_j} p_{Y/X}[y_j/x_i] = 1$$

eg: Two Die toss

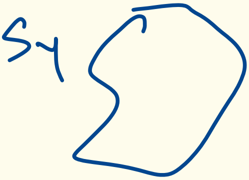
X, Y

$N_X = 6 \quad N_Y = 6$

$p_{Y/X} \rightarrow$ 6 PMFs each of length 6

$p_{X/Y} \rightarrow$ 6 PMFs each of length 6

Formulas



Conditional from Joint

1. $$p_{X/Y}[x_i/y_j] = \frac{p_{XY}[x_i, y_j]}{p_Y[y_j]}$$

$$p_{Y/X}[y_j/x_i] = \frac{p_{XY}[x_i, y_j]}{p_X[x_i]}$$

2. $$p_{X/Y}[x_i/y_j] = \frac{p_{XY}[x_i, y_j]}{\sum_{x_i} p_{XY}[x_i, y_j]}$$

$$p_{Y/X}[y_j/x_i] = \frac{p_{XY}[x_i, y_j]}{\sum_{y_j} p_{XY}[x_i, y_j]}$$

3. Joint from conditional

$$p_{XY}[x_i, y_j] = p_{X/Y}[x_i/y_j]\ p_Y[y_j]$$

$$p_{XY}[x_i, y_j] = p_{Y/X}[y_j/x_i]\ p_X[x_i]$$

4. Baye's Relation

$$p_{X/Y}\left[x_i/y_j\right] = \frac{p_{Y/X}\left[y_j/x_i\right] p_X\left[x_i\right]}{\sum_{x_i} p_{Y/X}\left[y_j/x_i\right] p_X\left[x_i\right]}$$

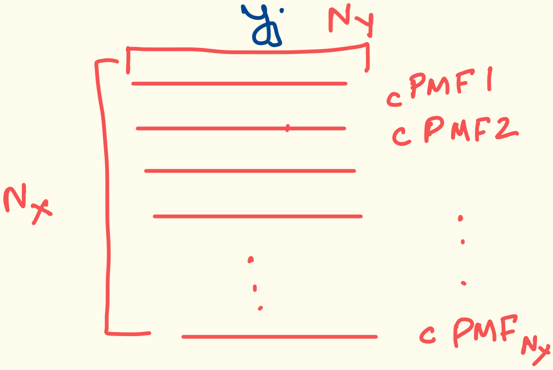
$$p_{Y/X}\left[y_j/x_i\right] = \frac{p_{X/Y}\left[x_i/y_j\right] p_Y\left[y_j\right]}{\sum_{y_j} p_{X/Y}\left[x_i/y_j\right] p_Y\left[y_j\right]}$$

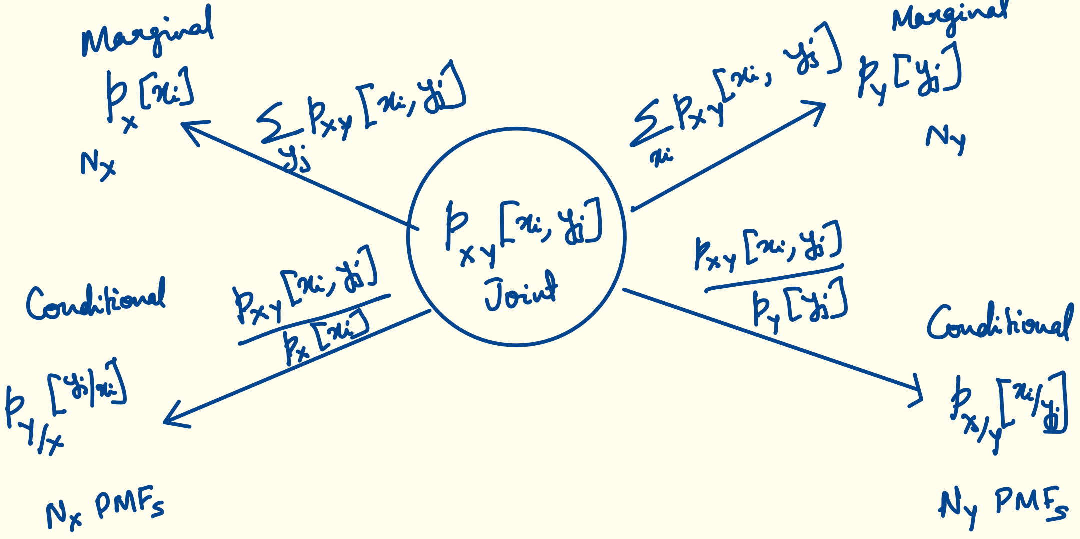
5. Marginal from Conditional

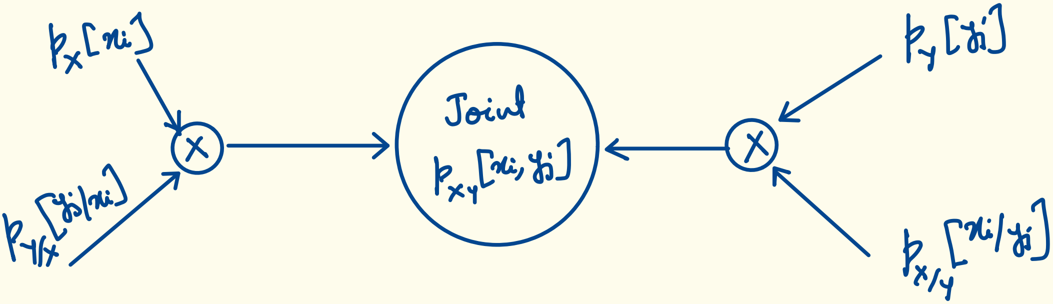
$$p_Y\left[y_j\right] = \sum_{x_i} p_{Y/X}\left[y_j/x_i\right] p_X\left[x_i\right]$$

$$p_X\left[x_i\right] = \sum_{y_j} p_{X/Y}\left[x_i/y_j\right] p_Y\left[y_j\right]$$

$p_{Y/X}\left[y_j/x_i\right]$







Conditional Expectation

$$E_{X/Y}[X/Y] = \sum_{x_i} x_i \; p_{X/Y}[x_i/y_j] \quad \leftarrow N_Y \text{ values}$$

$$E_{Y/X}[Y/X] = \sum_{y_j} y_j \; p_{Y/X}[y_j/x_i] \quad \leftarrow N_X \text{ values}$$

$E_{X/Y}[X/Y] \longrightarrow N_Y$ dimensional vector

$\longrightarrow$ function of Y

$E_{Y/X}[Y/X] \longrightarrow N_X$ dimensional vector

$\longrightarrow$ function of X

$$E_{X/Y}\left[g(X)/Y\right] = \sum_{x_i} g(x_i)\; p_{X/Y}\left[x_i/y_j\right] \leftarrow N_Y \text{ values}$$

$$E_{Y/X}\left[h(Y)/X\right] = \sum_{y_j} g(y_j)\; p_{Y/X}\left[y_j/x_i\right] \leftarrow N_X \text{ values}$$

$$g(X) = \left(X - E_{X/Y}[X/Y]\right)^2$$

$$Var\left(X/y_j\right) = \sum_{x_i} \left(x_i - E_{X/Y}\left[x_i/y_j\right]\right)^2 p_{X/Y}\left[x_i/y_j\right]$$

$y_j \rightarrow N_Y$ values

$$Var\left(Y/x_i\right) = \sum_{y_j} \left(y_j - E_{Y/X}\left[y_j/x_i\right]\right)^2 p_{Y/X}\left[y_j/x_i\right]$$

$x_i \rightarrow N_X$ values

$\mathrm{Var}(X/Y) \rightarrow N_Y$ dim vector

$\mathrm{Var}(Y/X) \rightarrow N_X$ dim vector

eg. $X \rightarrow \overset{H}{1}, \overset{T}{2}$ Coin toss

Die 1 $\rightarrow$ 1, 2, 3, 4, 5, 6

Fair

Die 2 $\rightarrow$ 2, 3, 2, 3, 2, 3

$$p_{Y/X}[y_j/1] = \begin{cases} \frac{1}{6}, & y_j = 1 \\ \frac{1}{6}, & y_j = 2 \\ \frac{1}{6}, & y_j = 3 \\ \frac{1}{6}, & y_j = 4 \\ \frac{1}{6}, & y_j = 5 \\ \frac{1}{6}, & y_j = 6 \end{cases}$$

$$p_{Y/X}[y_j/2] = \begin{cases} \frac{1}{2}, & y_j = 2 \\ \frac{1}{2}, & y_j = 3 \end{cases}$$

$$E_{Y/X}[Y/1] = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \cdots + 6 \cdot \frac{1}{6}$$

$$= \frac{1}{6}(1+2+3+4+5+6)$$

$$= \frac{21}{6}$$

$$= \frac{7}{2}$$

$$E_{Y/X}[Y/2] = \frac{1}{2}\cdot 2 + \frac{1}{2}\cdot 3$$

$$= 5/2$$

$$E_{Y/X}[Y/X] = \begin{bmatrix} 7/2 \\ 5/2 \end{bmatrix} \begin{matrix} x=1 \\ x=2 \end{matrix}$$ function of X, N_X dim. vector

$p_{X/Y}[x_i/y_j] \rightarrow$ function of X (N_X length)

$\rightarrow$ N_Y PMFs

$p_{Y/X}[y_j|x_i] \rightarrow$ function of Y (N_Y length)

$\rightarrow$ N_X PMFs

$\left.\begin{matrix} E_{X/Y}[X/Y] \\ \mathrm{var}(X/Y) \end{matrix}\right\} \rightarrow$ function of Y (N_Y length)

$\left.\begin{matrix} E_{Y/X}[Y/X] \\ \mathrm{var}(Y/X) \end{matrix}\right\} \rightarrow$ function of X (N_X length)

Unconditioning of Conditional Expectation

$E_{X/Y}[X/Y] \rightarrow N_Y$ dim vector

$\rightarrow$ function of Y

$h(Y) \rightarrow$ fn of Y

$$E_Y[h(Y)] = \sum_{y_i} h(y_i)\, p_Y[y_i]$$

$$E_Y\left[E_{X/Y}[X/Y]\right] = \sum_{y_i} E_{X/Y}[x_i/y_i]\, p_Y[y_i]$$

$$= \sum_{y_i} \sum_{x_i} x_i\, p_{X/Y}[x_i/y_i]\, p_Y[y_i]$$

$$= \sum_{x_i} x_i \sum_{y_i} p_{XY}[x_i, y_i]$$

$$= \sum_{x_i} x_i\, p_X[x_i]$$

$$= E_X[X]$$

$$E_y\left[E_{x/y}[x/y]\right] = E_x[x]$$

Unconditioning

$$E_x\left[E_{y/x}[y/x]\right] = E_y[y]$$

eg.

$$E_{y/x}[y/x] = \begin{array}{c} \begin{bmatrix} 7/2 \\ 5/2 \end{bmatrix} \end{array} \begin{array}{ll} x=1 & \frac{1}{2} \\ x=2 & \frac{1}{2} \end{array}$$

$$E_x\left[E_{y/x}[y/x]\right] = 7/2 \cdot \frac{1}{2} + \frac{5}{2} \cdot \frac{1}{2}$$

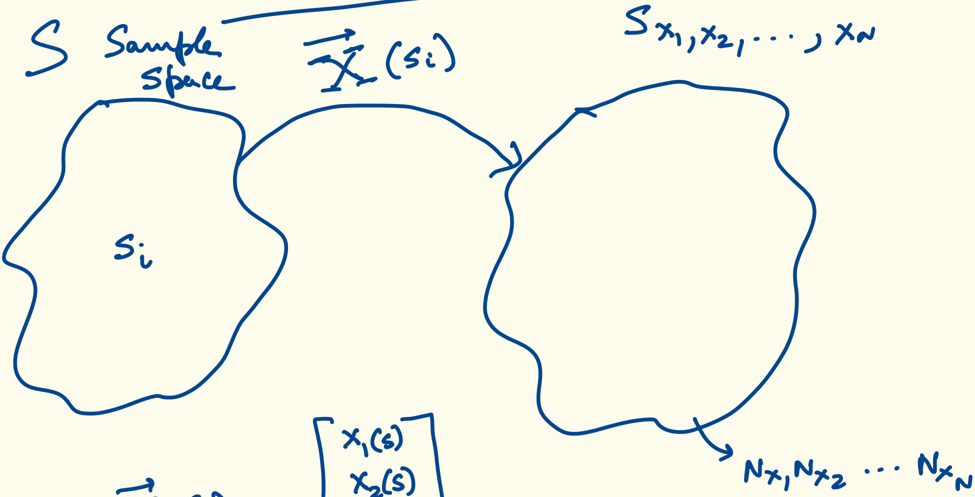
$$= \frac{12}{4}$$

$$= 3$$

$$E_y[y] = 3$$

Discrete N-D Random Variables

Random Vectors



$$\vec{X}(s) = \begin{bmatrix} X_1(s) \\ X_2(s) \\ X_3(s) \\ \vdots \\ X_N(s) \end{bmatrix}$$

$$\vec{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_N \end{bmatrix} \rightarrow \text{Random Vector}$$

$X_1, X_2, \ldots, X_N$ → Discrete R.Vs.

$\mathbb{Z} \quad \mathbb{Z} \quad \cdots \quad \mathbb{Z}$

$$\vec{X} \rightarrow \subset \mathbb{Z}^N$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_N \end{bmatrix} \rightarrow \text{One instance of the random vector } \vec{X}$$

$$S_{X_1} \rightarrow N_{X_1}$$

$$S_{X_2} \rightarrow N_{X_2}$$

$$\vdots$$

$$S_{X_N} \rightarrow N_{X_N}$$

$$S_{X_1, X_2, \ldots, X_N}$$

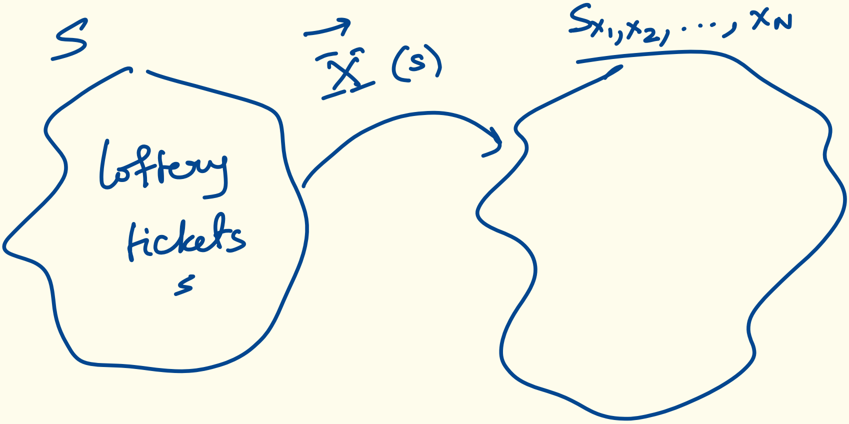
$$= S_{X_1} \times S_{X_2} \times \ldots \times S_{X_N}$$

Cartesian product

$$S_{X_1, X_2, \ldots, X_N}$$

$$\mapsto N_{X_1} N_{X_2} \ldots N_{X_N}$$

eg: 10 Digit lottery tickets (integers)



$$N_{X_1} \; N_{X_2} \; \ldots \; N_{X_N}$$

$$10 \quad 10 \quad \ldots \quad 10 \quad = 10^N \text{ tickets}$$

$$\vec{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{10} \end{bmatrix}$$

One instance (ticket)

$$\vec{X} = \vec{x} = \begin{bmatrix} 2 \\ 0 \\ 8 \\ 9 \\ 3 \\ 1 \\ 5 \\ 4 \\ 1 \\ 6 \end{bmatrix} \longrightarrow 10^N \text{ tickets}$$

Joint PMF

$$p_{X_1, X_2, \ldots, X_N}[x_1, x_2, \ldots, x_N] = P[X_1 = x_1, X_2 = x_2, \ldots, X_N = x_N]$$

$$p_{\vec{X}}[\vec{x}] = P[\vec{X} = \vec{x}]$$

$$\vec{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$$

$p_{\vec{X}}[\vec{x}] \rightarrow$ Assigns probability to all values in $S_{X_1, X_2, \ldots X_N}$

$(N_{X_1} N_{X_2} \cdots N_{X_N})$ $\quad S_{\vec{X}}$

$\vec{X}$ takes one out of $N_{X_1} N_{X_2} \cdots N_{X_N}$ values with a probability given by $p_{\vec{X}}[\vec{x}]$

Properties

1. $0 \leq p_{\vec{X}}[\vec{x}] \leq 1$

2. $$\sum_{\{x_1 : x_1 \in S_{X_1}\}} \sum_{\{x_2 : x_2 \in S_{X_2}\}} \cdots \sum_{\{x_N : x_N \in S_{X_N}\}} p_{\vec{X}}[\vec{x}] = 1$$

$N_{X_1} N_{X_2} \cdots N_{X_N}$ values

eg. Basket containing N different colored balls.

M balls are chosen at random

Probability that k_1 balls are of color 1

k_2 " " 2

$\vdots$

k_N " " N

Multinomial PMF

$$p_{X_1, X_2, \ldots, X_N}[X_1 = k_1, X_2 = k_2, \ldots, X_N = k_N]$$

$$= \frac{M!}{k_1!\, k_2! \ldots k_N!} p_1^{k_1} p_2^{k_2} \ldots p_N^{k_N}$$

$p_i \rightarrow$ Probability of drawing a ball with color i.

$$\sum_{i=1}^{N} k_i = M \qquad 0 \leq p_i \leq 1 \qquad \sum_{i=1}^{N} p_i = 1$$

$$M \geq k_i \geq 0$$

Marginal PMFs

$$p_{X_1}[k_1] = \sum_{k_2}\sum_{k_3}\cdots\sum_{k_N} p_{X_1,X_2,\ldots,X_N}[k_1,k_2,\ldots,k_N]$$

$$p_{X_1,X_N}[k_1,k_N] = \sum_{k_2}\sum_{k_3}\cdots\sum_{k_{N-1}} p_{X_1,X_2,\ldots,X_N}[k_1,k_2,\ldots,k_N]$$

$X_1, X_2, \ldots, X_N \rightarrow$ Independent

$$p_{X_1,X_2,\ldots,X_N}[k_1,k_2,\ldots,k_N] = p_{X_1}[k_1]\, p_{X_2}[k_2] \cdots p_{X_N}[k_N]$$

eg. N independent Bernoulli Trials

$$X_i \sim \mathrm{Ber}(p_i) \qquad i=1,2,\ldots,N$$

$$p_{X_i}[k_i] = p_i^{k_i}(1-p_i)^{(1-k_i)} \qquad k_i \in \{0,1\}$$

$$p_{X_1,X_2,\ldots,X_N}[k_1,k_2,\ldots,k_N] = \prod_{i=1}^{N} p_i^{k_i}(1-p_i)^{(1-k_i)}$$

$$= p_i^{\sum_{i=1}^{N} k_i}\,(1-p_i)^{\sum_{i=1}^{N}(1-k_i)}$$

$$p_{\vec{\underline{X}}}[\vec{k}] = p_i^{\sum_{i=1}^{N} k_i} (1-p_i)^{N - \sum_{i=1}^{N} k_i}$$

Joint Cumulative Distribution Function

(CDF)

$$F_{X_1, X_2, \cdots, X_N}(x_1, x_2, \cdots, x_N) = \sum_{k_1=-\infty}^{x_1} \sum_{k_2=-\infty}^{x_2} \cdots \sum_{k_N=-\infty}^{x_N} p_{\vec{\underline{X}}}[\vec{x}]$$

$$= P[X_1 \leq x_1, X_2 \leq x_2, \cdots, X_N \leq x_N]$$

$$F_{X_1, X_2, \cdots, X_N}(-\infty, -\infty, \cdots, -\infty) = 0$$

$$F_{X_1, X_2, \cdots, X_N}(\infty, \infty, \cdots, \infty) = 1$$

$$F_{X_1, X_2, \cdots, X_N}(x_1, \infty, \infty, \cdots, \infty) = F_{X_1}(x_1)$$

Transformation of Random Vector

$$\vec{\underline{X}} \rightarrow N \text{ dim. } \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$$

$$\vec{\underline{Y}} \rightarrow N \text{ dim. } \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}$$

$$\vec{\underline{Y}} = \vec{g}(\vec{\underline{X}})$$

$$y_1 = g_1(x_1, x_2, \ldots, x_N)$$

$$y_2 = g_2(x_1, x_2, \ldots, x_N)$$

$$\vdots \qquad \vdots$$

$$y_N = g_N(x_1, x_2, \ldots, x_N)$$

$\vec{g}^{-1}$ exists

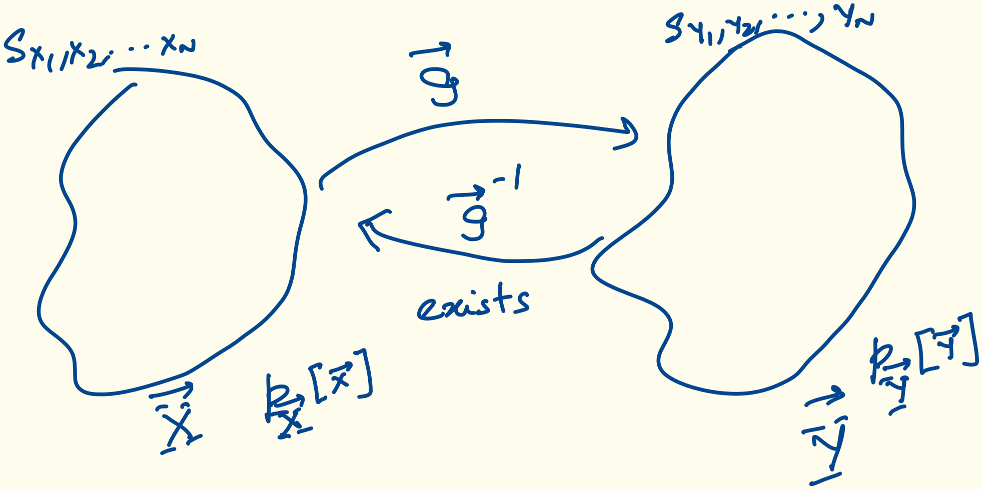
$$\vec{\underline{X}} = \vec{g}^{-1}(\vec{\underline{Y}})$$

$$x_1 = g_1^{-1}(y_1, y_2, \ldots, y_N)$$

$$x_2 = g_2^{-1}(y_1, y_2, \ldots, y_N)$$

$$\vdots$$

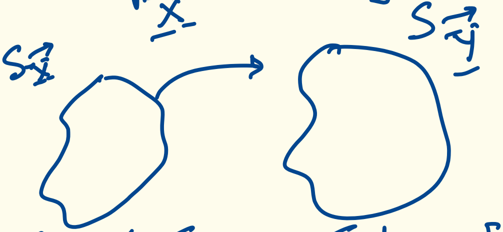
$$x_N = g_N^{-1}(y_1, y_2, \ldots, y_N)$$



One-one

$$p_{\vec{Y}}[\vec{y}] = p_{\vec{X}}\left[\vec{g}^{-1}\vec{y}\right]$$

Many - one



$$p_{Y_1,Y_2,\ldots,Y_N}[y_1,y_2,\ldots,y_N] = \sum\sum\cdots\cdots\sum_{\left\{\begin{array}{l}(x_1,x_2,\ldots,x_N): \\ y_1=g_1(x_1,x_2,\ldots,x_N) \\ y_2=g_2(x_1,x_2,\ldots,x_N) \\ \vdots \\ y_N=g_N(x_1,x_2,\ldots,x_N)\end{array}\right\}} p_{X_1,X_2,\ldots,X_N}[x_1,x_2,\ldots,x_N]$$

Linear Transformation of a Random Vector (Discrete)

Case 1 $\vec{Y}_{N\times 1} = A_{N\times N}\,\vec{X}_{N\times 1}$ $\quad \text{rank}(A) = N$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} = \underset{A}{\begin{bmatrix} \vec{a}_1^{\,T} \\ \vec{a}_2^{\,T} \\ \vdots \\ \vec{a}_N^{\,T} \end{bmatrix}} \underset{\vec{X}}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}}$$

$$\begin{aligned} y_1 &= \vec{a}_1^{\,T}\vec{X} & \qquad & g_1 \\ y_2 &= \vec{a}_2^{\,T}\vec{X} & & g_2 \\ &\vdots & & \vdots \\ y_N &= \vec{a}_N^{\,T}\vec{X} & & g_N \end{aligned}$$

$$A^{-1} = \begin{bmatrix} \vec{a}_1'^{\,T} \\ \vec{a}_2'^{\,T} \\ \vdots \\ \vec{a}_N'^{\,T} \end{bmatrix} \begin{matrix} g_1^{-1} \\ g_2^{-1} \\ \vdots \\ g_N^{-1} \end{matrix}$$

$$\vec{X} = A^{-1}\vec{Y}$$

$$p_{\vec{Y}}[\vec{y}] = p_{\vec{X}}[A^{-1}\vec{y}]$$

<u>Case 2</u>

$$\underset{M\times 1}{\vec{Y}} = \underset{M\times N}{A}\,\vec{X}_{N\times 1}\,, \quad M < N \quad \text{rank}(A) = M$$

$$\vec{Z} = \begin{bmatrix} \vec{Y} \\ \hline \text{from } \vec{X} \end{bmatrix} \begin{matrix} \}\, M\times 1 \\ \}\, N-M \end{matrix} \quad N\times 1$$

$$\tilde{A} = \begin{bmatrix} & A_{M\times N} & \\ \hline 0 \cdots 0 & 1 & 0\,0 \cdots 0 \\ & \vdots & \\ 0\,0 & \cdots 0\,\,0 & \cdots 1 \end{bmatrix} \begin{matrix} \\ \}\, N-M \text{ rows} \end{matrix} \quad N\times N \qquad \text{rank}(\tilde{A}) = N$$

$$\vec{Z}_1 = \tilde{A}\,\vec{X}$$

$$\vec{Z}_1 = \begin{bmatrix} \vec{Y}_{M\times 1} \\ \hline \text{from } \vec{X}_{N-M\times 1} \end{bmatrix}_{N\times 1} = \underset{\tilde{A}}{\begin{bmatrix} A_{M\times N} \\ \hline 0 \;\; 0 \;\; 0 \cdots 1 \;\; 0 \;\; 0 \cdot 0 \\ \vdots \\ 0 \;\; 0 \;\; 0 \;\; \cdots \;\; 0 \;\; 1 \end{bmatrix}_{N\times N}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}_{N\times 1}$$

$$\vec{X} = \tilde{A}^{-1}\vec{Z}_1$$

$$\text{rank}(\tilde{A}) = N$$

$$p_{\vec{Z}_1}[\vec{z}] = p_{\vec{X}}\left[\tilde{A}^{-1} z\right]$$

Marginalize the random variables

$$X_{M+1}, X_{M+2}, \ldots, X_N \qquad p_{\vec{Z}_1}[\vec{z}]$$

$$\Big\downarrow$$

$$p_{\vec{Y}}[\vec{Y}] = \sum_{x_{M+1}} \sum_{x_{M+2}} \cdots \sum_{x_N} p_{Y_1, Y_2, \ldots, Y_M, X_{M+1}, \ldots, X_N}\left[y_1, y_2, \ldots, y_M, x_{M+1} \ldots x_N\right]$$

eg: Three Independent Bernoulli Trials

$$p_{\vec{X}}[\vec{k}] = p^{(k_1+k_2+k_3)} (1-p)^{3-(k_1+k_2+k_3)}$$

$$\vec{k} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix}$$

$$\vec{X} = \begin{array}{cc} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} & \begin{array}{c} k_1 \\ k_2 \\ k_3 \end{array} \end{array}$$

$$\vec{Y} = \begin{array}{cc} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} & \begin{array}{c} l_1 \\ l_2 \\ l_3 \end{array} \end{array} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$l_1 = k_1$$

$$l_2 = k_1 + k_2$$

$$l_3 = k_1 + k_2 + k_3$$

$$k_1 = l_1$$

$$k_2 = l_2 - l_1$$

$$k_3 = l_3 - (k_2 + k_1)$$

$$= l_3 - l_2$$

$$p_{Y_1,Y_2,Y_3}[l_1,l_2,l_3] = p^{l_1+l_2-l_1+l_3-l_2}(1-p)^{3-(l_1+l_2-l_1+l_3-l_2)}$$

$$= p^{l_3}(1-p)^{3-l_3}$$

$l_3 \in \{0,1,2,3\}$

X_1	X_2	X_3	Y_1	Y_2	Y_3
k_1	k_2	k_3	l_1	l_2	l_3
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	1
0	1	1	0	1	2
1	0	0	1	1	1
1	0	1	1	1	2
1	1	0	1	2	2
1	1	1	1	2	3

eg. N independent Bernoulli trials

$$p_{\underline{\vec{X}}}[\vec{x}] = p^{\sum_{i=1}^{N} x_i} (1-p)^{N - \sum_{i=1}^{N} x_i}$$

$$Y = X_1 + X_2 + \cdots + X_N$$

$$Y = \underset{A}{\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}_{1\times N}} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix}_{N\times 1}$$

$$p_{X_i}[k_i] = p^{k_i} (1-p)^{1-k_i} , \quad k_i \in \{0, 1\}$$

$$\Phi_{X_i}(\omega) = \sum_{k_i=0}^{1} p^{k_i} (1-p)^{(1-k_i)} e^{j\omega k_i}$$

$$= p e^{j\omega} + (1-p)$$

\# of successes in N trials $\rightarrow Y = X_1 + X_2 + \cdots + X_N$ $\quad X_1, X_2, \ldots, X_N \rightarrow$ Independent

$$p_Y[y] = p_{X_1}[y] * p_{X_2}[y] * \cdots * p_{X_N}[y]$$

$$\Phi_Y(\omega) = \Phi_{X_1}(\omega) \cdot \Phi_{X_2}(\omega) \cdot \cdots \cdot \Phi_{X_N}(\omega)$$

$$\Phi_Y(\omega) = \left[p e^{j\omega} + (1-p)\right]^N$$

$$p_Y[y] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_Y(\omega)\, e^{-j\omega y}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left[\underset{a}{p e^{j\omega}} \underset{+}{+} \underset{b}{(1-p)}\right]^N e^{-j\omega y}\, d\omega$$

$$= \frac{1}{2\pi}\int_{\omega=-\pi}^{\pi} \sum_{z} \binom{N}{z} \left(p e^{j\omega}\right)^z (1-p)^{N-z} e^{-j\omega y}\, d\omega$$

$$= \frac{1}{2\pi}\int_{\omega=-\pi}^{\pi} \sum_{z} \binom{N}{z}\, p^z\, (1-p)^{N-z}\, e^{j\omega(z-y)}\, d\omega$$

$\cos \omega i + j \sin \omega i \quad z \neq y$

$$\int_{-\pi}^{\pi} e^{j\omega(z-y)}\, d\omega = \begin{cases} 1, & z=y \\ 0, & z\neq y \end{cases}$$

$$p_Y[y] = \binom{N}{y} p^y (1-p)^{N-y} \frac{1}{2\pi}\int_{\omega=-\pi}^{\pi} 1\, d\omega$$

$$p_Y[y] = \binom{N}{y} p^y (1-p)^{N-y} \longrightarrow$$ Binomial PMF

Expectation of a Discrete Random Vector

$\vec{X}_{N\times 1} \rightarrow$ Random Vector

$$E_{\vec{X}}[\vec{X}] = E_{\vec{X}}\begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix}$$

$$= \begin{bmatrix} E_{X_1}[X_1] \\ E_{X_2}[X_2] \\ \vdots \\ E_{X_N}[X_N] \end{bmatrix}_{N\times 1}$$

$$E_{\vec{X}}[X_i] = \sum_{x_1}\sum_{x_2}\cdots\sum_{x_N} x_i \, p_{\vec{X}}[x_1, x_2, \ldots, x_N]$$

$$= \sum_{x_i} x_i \, p_{X_i}[x_i] = E_{X_i}[X_i]$$

$$E_{\vec{X}}[g(X_1, X_2, \ldots, X_N)]$$

$$= \sum_{x_1}\sum_{x_2}\cdots\sum_{x_N} g(x_1, x_2, \ldots, x_N) \, p_{\vec{X}}[x_1, x_2, \ldots, x_N]$$

$$E_{\vec{X}}\left[\vec{a}^T\vec{X}\right] = E_{x_1,x_2,\cdots,x_N}\left[a_1x_1+a_2x_2+\cdots+a_Nx_N\right]$$

$$= a_1E_{x_1}[x_1]+a_2E_{x_2}[x_2]+\cdots+a_NE_{x_N}[x_N]$$

$$= \vec{a}^T\,E_{\vec{X}}\left[\vec{X}\right] = \sum_{i=1}^{N}a_iE_{x_i}[x_i]$$

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}, \quad E_{\vec{X}}\left[\vec{X}\right] = \begin{bmatrix} E_{x_1}[x_1] \\ E_{x_2}[x_2] \\ \vdots \\ E_{x_N}[x_N] \end{bmatrix}$$

$$\mathbb{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{N\times 1}$$

$$E_{\vec{X}}\left[\mathbb{1}^T\vec{X}\right] = E_{\vec{X}}\left[x_1+x_2+\cdots+x_N\right]$$

$$= E_{x_1}[x_1]+E_{x_2}[x_2]+\cdots+E_{x_N}[x_N]$$

$$= \mathbb{1}^T\,E_{\vec{X}}\left[\vec{X}\right]$$

$$= \sum_{i=1}^{N}E_{x_i}[x_i]$$

$$\mathrm{Var}\left(\mathbb{1}^T \vec{X}\right) = \mathrm{Var}\left(\underbrace{\sum_{i=1}^{N} x_i}_{Y}\right) \longrightarrow ①$$

$$E_{\vec{X}}\left[g(x_1, x_2, \ldots, x_N)\right] = E_Y\left[\left(Y - E_Y[Y]\right)^2\right]$$

$$= E_{\vec{X}}\left[\left(\sum_{i=1}^{N} x_i - E_{\vec{X}}\left(\sum_{i=1}^{N} x_i\right)\right)^2\right] \longrightarrow ②$$

From ① & ②

$$\mathrm{Var}\left(\sum_{i=1}^{N} x_i\right) = E_{\vec{X}}\left[\left(\sum_{i=1}^{N} x_i - \sum_{i=1}^{N} E_{x_i}[x_i]\right)^2\right]$$

$$= E_{\vec{X}}\left[\left(\sum_{i=1}^{N} \left(x_i - E_{x_i}[x_i]\right)\right)^2\right]$$

$$U_i = x_i - E_{x_i}[x_i]$$

$$\sum_{i=1}^{N} \left(x_i - E_{x_i}[x_i]\right)^2 = \left(\sum_{i=1}^{N} U_i\right)^2$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} U_i U_j$$

$$\mathrm{Var}\left(\sum_{i=1}^{N} x_i\right) = E_{\underline{\vec{X}}}\left(\sum_{i=1}^{N}\sum_{j=1}^{N}(x_i - E_{x_i}[x_i])(x_j - E_{x_j}[x_j])\right)$$

$$= \sum_{i=1}^{N}\sum_{j=1}^{N} E_{\underline{\vec{X}}}\left[(x_i - E_{x_i}[x_i])(x_j - E_{x_j}[x_j])\right]$$

$$\mathrm{Var}\left(\sum_{i=1}^{N} x_i\right) = \underbrace{\sum_{i=1}^{N}\sum_{j=1}^{N} E_{x_i x_j}\left[(x_i - E_{x_i}[x_i])(x_j - E_{x_j}[x_j])\right]}_{N^2}$$

$$= \sum_{i=1}^{N} E_{x_i}\left[(x_i - E_{x_i}[x_i])^2\right] + \sum_{\substack{i=1 \\ i \neq j}}^{N}\sum_{j=1}^{N} E_{x_i x_j}\left[(x_i - E_{x_i}[x_i])(x_j - E_{x_j}[x_j])\right]$$

$$= \sum_{i=1}^{N} \mathrm{Var}(x_i) + \sum_{\substack{i=1 \\ i \neq j}}^{N}\sum_{j=1}^{N} \mathrm{Cov}(x_i, x_j)$$

$$\mathrm{Var}\left(\sum_{i=1}^{N} x_i\right) = \mathrm{Var}\left(\mathbb{1}^T \underline{\vec{X}}\right)$$

$$= \mathbb{1}^T C_{\underline{\vec{X}}} \mathbb{1}$$

$$= \underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}}_{\mathbb{1}^T}{}_{1 \times N} \underbrace{\begin{bmatrix} \mathrm{var}(x_1) & \mathrm{cov}(x_1, x_2) & \cdots & \mathrm{cov}(x_1, x_N) \\ \mathrm{cov}(x_2, x_1) & \mathrm{var}(x_2) & & \vdots \\ \vdots & & \ddots & \vdots \\ \mathrm{cov}(x_N, x_1) & \cdots & & \mathrm{var}(x_N) \end{bmatrix}}_{C_{\vec{X}}}{}_{N \times N} \underbrace{\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}}_{\mathbb{1}}{}_{N \times 1}$$

Covariance Matrix

$$C_{\vec{X}} = \begin{bmatrix} \mathrm{var}(x_1) & \mathrm{cov}(x_1, x_2) & \cdots & \mathrm{cov}(x_1, x_N) \\ \mathrm{cov}(x_2, x_1) & \mathrm{var}(x_2) & \cdots & \mathrm{cov}(x_2, x_N) \\ \vdots & & \ddots & \vdots \\ \mathrm{cov}(x_N, x_1) & \cdots & & \mathrm{var}(x_N) \end{bmatrix}_{N \times N}$$

Properties of $C_{\vec{X}}$ (Covariance Matrix)

1. $C_{\vec{X}}$ is symmetric.

Proof For any 2 r.v.s. x_i, x_j $i \neq j$

$$\mathrm{Cov}(x_i, x_j) = E_{x_i x_j}\left[(x_i - E_{x_i}[x_i])(x_j - E_{x_j}[x_j])\right]$$

$$= E_{x_j x_i}\left[(x_j - E_{x_j}[x_j])(x_i - E_{x_i}[x_i])\right]$$

$$= \mathrm{Cov}(x_j, x_i)$$

$\Rightarrow$ $C_{\vec{X}}$ is symmetric.

2. $C_{\vec{X}}$ is positive semidefinite. $\forall \vec{a} \in \mathbb{R}^N$ $\vec{a}^T C_{\vec{X}} \vec{a} \geq 0$

Proof $\mathrm{var}(\mathbb{1}^T \vec{X}) = \mathbb{1}^T C_{\vec{X}} \mathbb{1}$

$$\underset{\substack{\vee\!| \\ 0}}{\mathrm{var}(\vec{a}^T \vec{X})} = \vec{a}^T C_{\vec{X}} \vec{a} \geq 0$$

$\square$

3. $\underset{N\times 1}{\underline{\vec{Y}}} = \underset{N\times N}{\underline{A^T}}\ \underset{N\times 1}{\underline{\vec{X}}}$ Linear Transformatio

Rank $(A^T) = N$

$$C_{\underline{\vec{Y}}} = A^T C_{\underline{\vec{X}}} A$$

<u>Proof</u>

$$\underset{\underline{\vec{Y}}}{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}} = \underset{A^T}{\begin{bmatrix} \vec{a}_1^{\,T} \\ \vec{a}_2^{\,T} \\ \vdots \\ \vec{a}_N^{\,T} \end{bmatrix}} \underset{\underline{\vec{X}}}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}}$$

from property 2

$$C_{\underline{\vec{Y}}} = A^T C_{\underline{\vec{X}}} A$$

4. If $\vec{\underline{X}}$ is uncorrelated, $C_{\vec{\underline{X}}}$ is diagonal.

Proof As x_i, x_j $i \neq j$ are uncorrelated,

$$\operatorname{Cov}(x_i, x_j) = 0 \quad i \neq j$$

$$C_{\vec{\underline{X}}} = \begin{bmatrix} \operatorname{var}(x_1) & & & 0 \\ & \operatorname{var}(x_2) & & \\ & & \ddots & \\ 0 & & & \operatorname{var}(x_N) \end{bmatrix}$$

$\square$

5. $C_{\vec{\underline{X}}}$ is always diagonalizable.

Proof $C_{\vec{\underline{X}}}$ is symmetric $\Rightarrow$ Eigenvectors are real and orthogonal. normal

$C_{\vec{\underline{X}}}$ is +ve semidefinite $\Rightarrow$ Eigen values are non-negative

$\lambda_i \geq 0$

$$C_{\vec{\underline{X}}} \vec{v}_i = \lambda_i \vec{v}_i \, , \quad i = 1, 2, \ldots, N$$

$$\vec{v}_i^{\,T} \vec{v}_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} , \quad \vec{v}_i \in \mathbb{R}^N$$

$$C_{\vec{X}} \underbrace{\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_N \end{bmatrix}}_{V} = \underbrace{\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_N \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_N \end{bmatrix}}_{\Lambda}$$

$$V V^T = V^T V = I$$

$$C_{\vec{X}} V = V \Lambda$$

$$C_{\vec{X}} \underbrace{V V^T}_{I} = V \Lambda V^T$$

$$C_{\vec{X}} = \underbrace{V}_{\substack{\text{eigenvectors} \\ \text{of } C_{\vec{X}} \\ \text{as cols}}} \overset{\substack{\text{eigenvalues} \\ \text{of } C_{\vec{X}} \text{ on diagonal} \\ \downarrow}}{\Lambda} \underbrace{V^T}_{\substack{\text{eigenvectors} \\ \text{of } C_{\vec{X}} \\ \text{as rows}}}$$

$C_{\vec{X}}$ is diagonalizable.

Trefethen and Bau

Numerical Linear Algebra

$$C_{\vec{\underline{X}}} = E_{\vec{\underline{X}}}\left[(\vec{\underline{X}}-\vec{\mu})(\vec{\underline{X}}-\vec{\mu})^T\right]$$

$$\vec{\mu} = E_{\vec{\underline{X}}}\left[\vec{\underline{X}}\right]$$

$$\begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \\ \vdots \\ x_N-\mu_N \end{bmatrix} \begin{bmatrix} x_1-\mu_1 & x_2-\mu_2 & \cdots & x_N-\mu_N \end{bmatrix}$$

$$= \begin{bmatrix} (x_1-\mu_1)^2 & (x_1-\mu_1)(x_2-\mu_2) & \cdots & (x_1-\mu_1)(x_N-\mu_N) \\ (x_1-\mu_1)(x_2-\mu_2) & (x_2-\mu_2)^2 & & \vdots \\ \vdots & & \ddots & \vdots \\ (x_1-\mu_1)(x_N-\mu_N) & \cdots & & (x_N-\mu_N)^2 \end{bmatrix}$$

Dimensionality Reduction

$\vec{\underline{X}} \rightarrow$ Data vectors $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}$

$$\vec{x}_i \in \mathbb{R}^m$$

$\vec{\underline{Y}} \rightarrow$ Reduced Data vectors $\{\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_n\}$

$$\vec{y}_i \in \mathbb{R}^k \qquad k << m$$

Data Matrix

$$D_x = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \\ & & & \end{bmatrix}_{m \times n}$$

$k << m$

Reduced Data Matrix

$$D_y = \begin{bmatrix} \vec{y}_1 & \vec{y}_2 & \cdots & \vec{y}_n \\ & & & \end{bmatrix}_{k \times n}$$

$$C_{\vec{\underline{X}}} = E_{\vec{\underline{X}}}\left[(\vec{\underline{X}} - \vec{\mu})(\vec{\underline{X}} - \vec{\mu})^T \right]$$

Principal Component Analysis

Given $D_x = \left[\vec{x}_1 \ \vec{x}_2 \ \dots \ \vec{x}_n\right]_{m \times n}$

To find $D_y = \left[\vec{y}_1 \ \vec{y}_2 \ \dots \ \vec{y}_n\right]_{k \times n}$

1. Compute the mean vector.

$$\vec{q} = \frac{1}{n} \sum_{i=1}^{n} \vec{x}_i$$

2. Subtract $\vec{q}$ from each data vector.

$$\vec{x}_i' = \vec{x}_i - \vec{q}$$

3. Form the matrix X with $\vec{x}_i'$ as columns

$$X = \begin{bmatrix} \vec{x}_1' & \vec{x}_2' & \dots & \vec{x}_n' \\ \\ \end{bmatrix}_{m \times n}$$

4. Compute Sample Covariance Matrix.

$$C_{x_{m \times m}} = \frac{1}{n} X_{m \times n} X^T_{n \times m}$$

5. Diagonalize C_x

$$\underset{m\times m}{C_x} = \underset{m\times m}{V}\ \underset{m\times m}{\Lambda}\ \underset{m\times m}{V^T}$$

$\Lambda \rightarrow$ diagonal eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m$

$V^T \rightarrow$ contains eigenvectors of C_x as rows

$$V^T V = V V^T = I$$

6. Estimate $V^{T'}$ (P) from V^T by retaining only top 'k' rows $(\lambda_1, \lambda_2, \dots, \lambda_k)$.

$$V^{T'} = P = \begin{bmatrix} \vec{v}_1^T \\ \vec{v}_2^T \\ \vdots \\ \vec{v}_k^T \end{bmatrix}_{k\times m}$$

$\vec{v}_i^T \rightarrow$ Principal components

7. Project each point $\vec{x}_i$ using P

$$\underset{k\times 1}{\vec{y}_i} = \underset{k\times m}{P}\ \underset{m\times 1}{\vec{x}_i}, \quad i = 1, 2, \dots, n$$

8. Reduced Data matrix

$$D_y = \begin{bmatrix} \vec{y}_1 & \vec{y}_2 & \dots & \vec{y}_n \end{bmatrix}_{k\times n}$$

$\vec{y}_i \in R^k$

eg.

$m = 10^3 , \quad n = 10^6$

$D_X \rightarrow 10^3 \times 10^6 \sim 10^9$

$D_Y \rightarrow 10 \times 10^6 \sim 10^7$

Higher Order Moments

$$E_{\vec{X}}\left[X_1^{l_1} X_2^{l_2} \dots X_N^{l_N}\right] = \sum_{x_1} \sum_{x_2} \dots \sum_{x_N} x_1^{l_1} x_2^{l_2} \dots x_N^{l_N} \, p_{\vec{X}}[x_1, x_2, \dots, x_N]$$

If $X_1, X_2, \dots, X_N$ are independent,

$$p_{X_1, X_2, \dots, X_N}[x_1, x_2, \dots, x_N] = p_{X_1}[x_1] \, p_{X_2}[x_2] \dots p_{X_N}[x_N]$$

$$E_{X_1, X_2, \dots, X_N}\left[X_1^{l_1} X_2^{l_2} \dots X_N^{l_N}\right] = E_{X_1}\left[X_1^{l_1}\right] E_{X_2}\left[X_2^{l_2}\right] \dots E_{X_N}\left[X_N^{l_N}\right]$$

Joint Characteristic Function

$$\Phi_{\vec{X}}(\omega_1, \omega_2, \dots, \omega_N) = E_{\vec{X}}\left[e^{j(\omega_1 X_1 + \omega_2 X_2 + \dots + \omega_N X_N)}\right]$$

$$= \sum_{x_1} \sum_{x_2} \cdots \sum_{x_N} e^{j(\omega_1 x_1 + \cdots + \omega_N x_N)} \quad p_{\underline{\vec{X}}}[x_1, x_2, \ldots, x_N]$$

N-D DTFT

If $x_1, x_2, \ldots, x_N$ are independent,

$$p_{x_1, x_2, \ldots, x_N}[x_1, x_2, \ldots, x_N] = p_{x_1}[x_1]\, p_{x_2}[x_2] \cdots p_{x_N}[x_N]$$

$$\underline{\Phi}_{x_1, x_2, \ldots, x_N}(\omega_1, \omega_2, \ldots, \omega_N) = \underline{\Phi}_{x_1}(\omega_1)\, \underline{\Phi}_{x_2}(\omega_2) \cdots \underline{\Phi}_{x_N}(\omega_N)$$

ND IDTFT

$$p_{\underline{\vec{X}}}[x_1, x_2, \ldots, x_N]$$

$$= \frac{1}{(2\pi)^N} \int_{\omega_1 = -\pi}^{\pi} \cdots \int_{\omega_N = -\pi}^{\pi} \underline{\Phi}_{x_1, x_2, \ldots, x_N}(\omega_1, \omega_2, \ldots, \omega_N)\, e^{-j(\omega_1 x_1 + \cdots + \omega_N x_N)}\, d\omega_1 \cdots d\omega_N$$

$$E_{\underline{\vec{X}}}[x_1^{l_1} x_2^{l_2} \cdots x_N^{l_N}] = \frac{1}{j^{l_1 + \cdots + l_N}} \frac{\partial^{l_1 + \cdots + l_N}}{\partial \omega_1^{l_1} \cdots \partial \omega_N^{l_N}} \underline{\Phi}_{\underline{\vec{X}}}(\omega_1, \ldots, \omega_N) \Bigg|_{\substack{\omega_1 = 0 \\ \omega_2 = 0 \\ \vdots \\ \omega_N = 0}}$$

Conditional PMFs

$$\vec{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix}$$

$$p_{X_N/X_1,X_2,\ldots,X_{N-1}}\left[\overset{A}{x_N}/\overset{B}{x_1,\ldots x_{N-1}}\right] = \frac{p_{X_1,X_2,\ldots,X_N}\overset{A\cap B}{[x_1,x_2,\ldots,x_N]}}{p_{X_1,X_2,\ldots,X_{N-1}}[x_1,x_2,\ldots,x_{N-1}]}$$

$$p_{X_1,\ldots,X_N} = p_{X_N/X_1,\ldots X_{N-1}}\, p_{X_1,\ldots,X_{N-1}} \quad — ①$$

$$p_{X_1,\ldots,X_{N-1}} = p_{X_{N-1}/X_1,\ldots,X_{N-2}}\, p_{X_1,X_2,\ldots X_{N-2}} \quad — ②$$

② in $\boxed{1}$

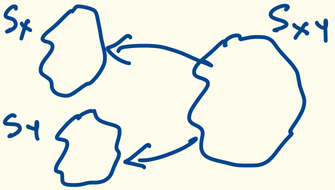
$$p_{X_1,\ldots,X_N} = p_{X_N/X_1,\ldots,X_{N-1}}\, p_{X_{N-1}/X_1,\ldots,X_{N-2}}\, p_{X_1,\ldots,X_{N-2}}$$

$$p_{X_1,\ldots,X_N} = p_{X_N/X_1,\ldots,X_{N-1}}\, p_{X_{N-1}/X_1,\ldots,X_{N-2}}\, p_{X_{N-2}/X_1,\ldots,X_{N-3}}$$

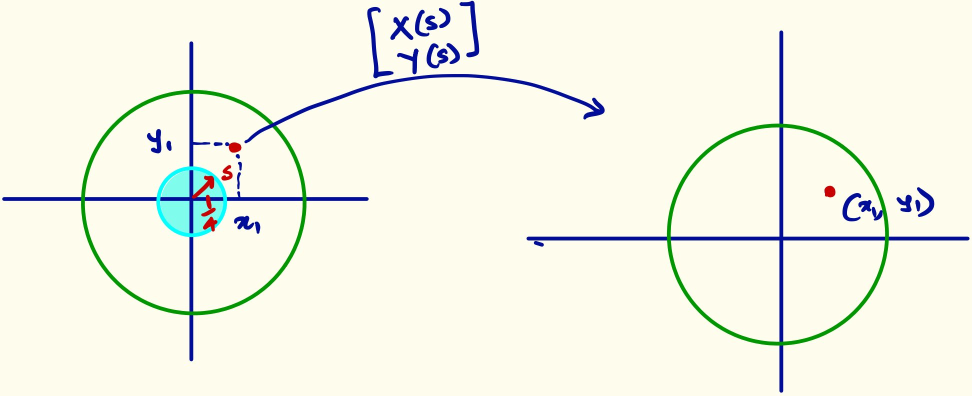
$$\ldots\ p_{X_2/X_1}\, p_{X_1}$$

Two Continuous Random Variables

$\begin{bmatrix} X \\ Y \end{bmatrix} \in \mathbb{R}^2$



eg: Dart board with radius 1



Joint Probability Density Function

$p_{X,Y}(x,y) \longrightarrow$ Function on 2D (surface)

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{\infty} p_{X,Y}(x,y)\,dx\,dy = 1$$

$\longrightarrow$ Volume under the surface is equal to 1.

eg:

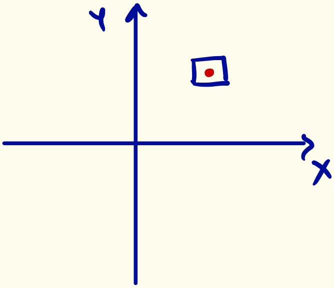
$$p_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi}, & x^2+y^2 \leq 1 \\ 0, & \text{o/w} \end{cases}$$

A → Dart hits the bullseye.

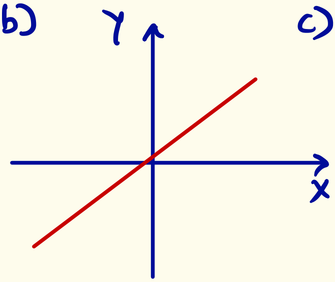
$$P[A] = \iint\limits_{x^2+y^2\le\frac{1}{4}} \frac{1}{\pi}\,dx\,dy = \frac{1}{\pi}\,\frac{\pi}{16} = \frac{1}{16}$$

$p_{X,Y}(x,y) = 0$

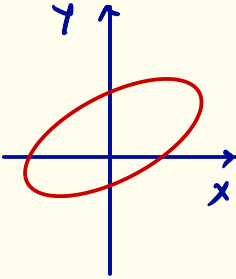
a)



b)



c)



$\underline{p_{X,Y}(x,y)} = 0$ for any curve in the X-Y plane.

as it does not enclose any volume.

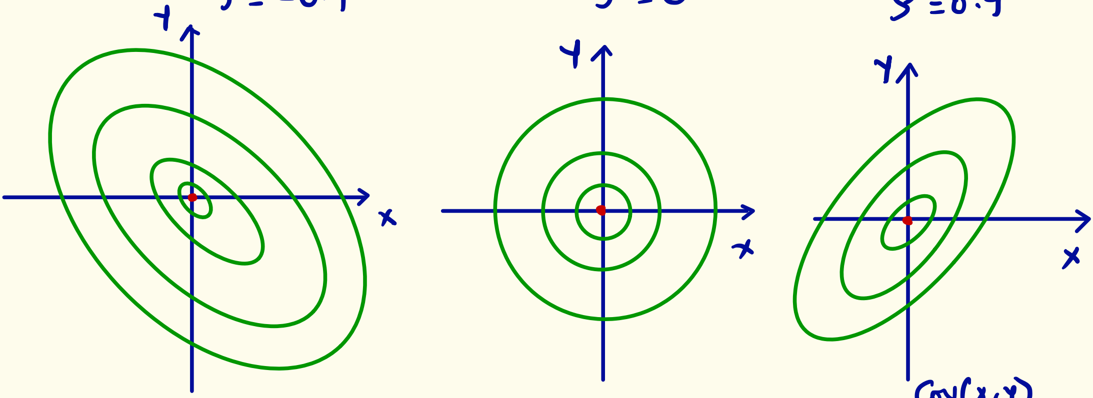
eg. Joint Gaussian PDF

$$p_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}(x^2-2\rho xy+y^2)} , \quad -\infty < x < \infty \quad -\infty < y < \infty$$

Bivariate Gaussian / Normal PDF

$\rho \rightarrow$ Correlation Coefficient $\quad -1 \leq \rho \leq 1$

$\rho = -0.9$ $\quad$ $\rho = 0$ $\quad$ $\rho = 0.9$



$$\rho = \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$$

$$p_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2}\frac{r^2}{(1-\rho^2)}}$$

$$r^2 = x^2-2\rho xy+y^2$$

$$x^2-2\rho xy+y^2 = \begin{bmatrix} x & y \end{bmatrix} \underbrace{\begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}}_{\substack{\text{Symmetric} \\ \text{+ve definite}}} \begin{bmatrix} x \\ y \end{bmatrix} > 0$$

Marginal PDFs

$$p_X(x) = \int_y p_{X,Y}(x,y)\,dy$$

$$p_Y(y) = \int_x p_{X,Y}(x,y)\,dx$$

eg.

$$p_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}}\, e^{-\frac{1}{2(1-\rho^2)}(x^2-2\rho xy+y^2)}$$

$$p_X(x) = \int_{y=-\infty}^{+\infty} \frac{1}{2\pi\sqrt{1-\rho^2}}\, e^{-\frac{1}{2(1-\rho^2)}(x^2-2\rho xy+y^2)}\,dy$$

$$x^2 \underbrace{-2\rho xy}_{-2ab} \underbrace{+y^2}_{a^2} + \underbrace{\rho^2x^2}_{b^2} - \rho^2x^2 = (\underset{a\ -\ b}{y-\rho x})^2 + x^2(1-\rho^2)$$

$$p_X(x) = \int_{y=-\infty}^{+\infty} \frac{1}{2\pi\sqrt{1-\rho^2}}\, e^{-\frac{1}{2(1-\rho^2)}((y-\rho x)^2)}\, \boxed{e^{-\frac{1}{2(1-\rho^2)}x^2(1-\rho^2)}}\,dy$$

$$\mu_y = \rho x, \quad \sigma_y^2 = 1-\rho^2$$

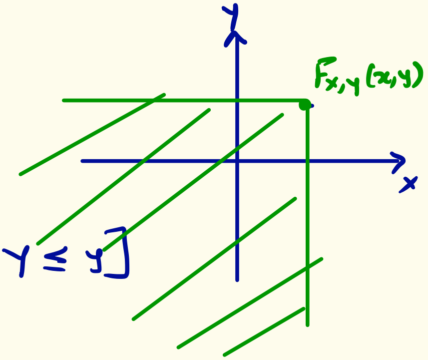
$$p_X(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2(1-\rho^2)}x^2(1-\rho^2)} \underset{\substack{\shortparallel \\ 1}}{\int_{y=-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\,\sigma_y}\, e^{-\frac{1}{2\sigma_y^2}(y-\mu_y)^2}\,dy}$$

$$p_x(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2\cancel{(1-\rho^2)}} x^2 \cancel{(1-\rho^2)}} \qquad \rho \neq 1$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} x^2} \quad , \quad -\infty < x < \infty$$

$X \sim N(0,1)$

$$p_y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} y^2} \quad , \quad -\infty < y < \infty \quad Y \sim N(0,1)$$



Joint CDF

$$F_{x,y}(x,y) = P[X \le x, \ Y \le y]$$

$$= \int_{-\infty}^{x} \int_{-\infty}^{y} p_{x,y}(x', y') \, dx' \, dy'$$

$$p_{x,y}(x,y) = \frac{\partial^2}{\partial x \, \partial y} F_{x,y}(x,y)$$

eg. Joint Exponential PDF

$$p_{X,Y}(x,y) = \begin{cases} e^{-(x+y)} , & x \geqslant 0 , \ y \geqslant 0 \\ 0 , & o/w \end{cases}$$

$$F_{X,Y}(x,y) = \int_0^x \int_0^y e^{-(t+u)} \, dt \, du$$

$$= \int_0^x e^{-t} dt \int_0^y e^{-u} du = \frac{e^{-t}}{-1}\Bigg|_0^x - \frac{e^{-u}}{-1}\Bigg|_0^y$$

$$= \left[e^{-x} - 1 \right] \left[e^{-y} - 1 \right] \qquad x \geqslant 0 , \ y \geqslant 0$$

$$p_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} \left[e^{-x} - 1 \right] \left[e^{-y} - 1 \right]$$

$$= \frac{\partial}{\partial x} - e^{-y} \left[e^{-x} - 1 \right]$$

$$= e^{-(x+y)} \qquad x \geqslant 0 , y \geqslant 0$$

$$F_{XY}(-\infty , -\infty) = 0$$

$$F_{XY}(\infty , \infty) = 1$$

Independence

$X, Y \rightarrow$ Independent iff

$$p_{X,Y}(x,y) = p_X(x)\; p_Y(y)$$

$$F_{X,Y}(x,y) = F_X(x)\; F_Y(y)$$

eg. $$p_{X,Y}(x,y) = e^{-x} e^{-y} = p_X(x)\; p_Y(y)$$

$X, Y \rightarrow$ Uncorrelated

eg. If $\rho = 0$ Bivariate Gaussian

$$p_{X,Y}(x,y) = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}, \quad -\infty < x < \infty, \; -\infty < y < \infty$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}$$

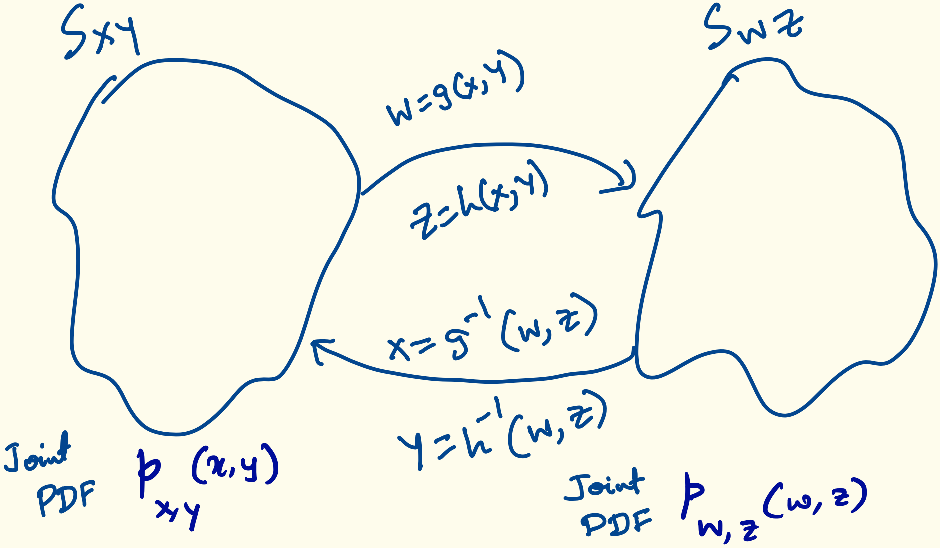
$$= p_X(x) \qquad p_Y(y)$$

For Bivariate Gaussian

$\rho = 0 \Rightarrow X, Y$ are uncorrelated $\quad E_{X,Y}[XY] = E_X[X]\; E_Y[Y]$

$\Rightarrow X, Y$ are independent.

Transformation of Random Variables



$W = g(x,y)$ $Z = h(x,y)$

$x = g^{-1}(w,z)$ as $w = g(x,y)$

$y = h^{-1}(w,z)$ as $z = h(x,y)$

General Transformation

$x = g^{-1}(w,z) \quad y = h^{-1}(w,z)$

$$p_{w,z}(w,z) = p_{x,y}\left(g^{-1}(w,z),\ h^{-1}(w,z)\right) \left| \underbrace{\det \frac{\partial(x,y)}{\partial(w,z)}}_{J^{-1}} \right|$$

$$\text{Jacobian} \quad J^{-1} = \frac{\partial(x,y)}{\partial(w,z)} = \begin{bmatrix} \frac{\partial x}{\partial w} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial w} & \frac{\partial y}{\partial z} \end{bmatrix}$$

Linear Transformation

$$\begin{bmatrix} W \\ Z \end{bmatrix} = \underset{\uparrow \text{ rank}=2}{G_{2\times2}} \begin{bmatrix} X \\ Y \end{bmatrix}$$

first row of $G \rightarrow g$

second row of $G \rightarrow h$

$G \rightarrow$ invertible

$$p_{W,Z}(w,z) = p_{X,Y}\left(G^{-1}\begin{bmatrix} w \\ z \end{bmatrix}\right) \left| \det(G^{-1}) \right| \leftarrow \text{ratio of areas in } X\text{-}Y \text{ \& } W\text{-}Z$$

$X, Y \sim N(0,1)$

eg. $X, Y \rightarrow$ standard Bivariate Gaussian

$$p_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[x^2 - 2\rho xy + y^2\right]}$$

$$\begin{bmatrix} W \\ Z \end{bmatrix} = \begin{bmatrix} \sigma_w & 0 \\ 0 & \sigma_z \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

$X \sim N(0,1)$

$Y \sim N(0,1)$

To show $W \sim N(0, \sigma_w^2)$, $Z \sim N(0, \sigma_z^2)$

$$p_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}}\, e^{-\frac{1}{2(1-\rho^2)}\left[x^2 - 2\rho xy + y^2\right]}$$

$$= \frac{1}{2\pi\sqrt{1-\rho^2}}\, e^{-\frac{1}{2(1-\rho^2)}\begin{bmatrix}x & y\end{bmatrix}\begin{bmatrix}1 & -\rho\\ -\rho & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}}$$

Covariance Matrix $C_{XY} = \begin{bmatrix}1 & \rho\\ \rho & 1\end{bmatrix}$

$$\det(C_{XY}) = 1-\rho^2$$

$$C_{XY}^{-1} = \frac{1}{1-\rho^2}\begin{bmatrix}1 & -\rho\\ -\rho & 1\end{bmatrix}$$

$$\Rightarrow \quad p_{XY}(x,y) = \frac{1}{2\pi \det^{\frac{1}{2}}(C_{XY})}\, e^{-\frac{1}{2}\begin{bmatrix}x & y\end{bmatrix} C_{XY}^{-1}\begin{bmatrix}x\\ y\end{bmatrix}}$$

$$\begin{bmatrix}X\\ Y\end{bmatrix} \sim \mathcal{N}\Big(\underset{\text{Mean Vector}}{\underset{2\times1}{\vec{0}}},\ \underset{\text{Covariance Matrix}}{\underset{2\times2}{C_{XY}}}\Big) \rightarrow \begin{bmatrix}1 & \rho\\ \rho & 1\end{bmatrix}$$

$$G = \begin{bmatrix} \sigma_w & 0 \\ 0 & \sigma_z \end{bmatrix} \qquad G^{-1} = \begin{bmatrix} \frac{1}{\sigma_w} & 0 \\ 0 & \frac{1}{\sigma_z} \end{bmatrix} \qquad \begin{bmatrix} w \\ z \end{bmatrix} = \begin{bmatrix} \sigma_w & 0 \\ 0 & \sigma_z \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\det G^{-1} = \frac{1}{\sigma_w \sigma_z}$$

$$w = \sigma_w x \qquad x = \frac{w}{\sigma_w}$$

$$z = \sigma_z y \qquad y = \frac{z}{\sigma_z}$$

$$p_{w,z}(w,z) = p_{x,y}\left(G^{-1} \begin{bmatrix} w \\ z \end{bmatrix} \right) \left| \det G^{-1} \right|$$

$$= \frac{1}{2\pi \sqrt{(1-\rho^2)\sigma_w^2 \sigma_z^2}} e^{\frac{1}{2(1-\rho^2)}\left[\left(\frac{w}{\sigma_w}\right)^2 - \frac{2\rho w z}{\sigma_w \sigma_z} + \left(\frac{z}{\sigma_z}\right)^2 \right]}$$

$$C_{w,z} = \begin{bmatrix} \sigma_w^2 & \rho\sigma_w\sigma_z \\ \rho\sigma_w\sigma_z & \sigma_z^2 \end{bmatrix}$$

$$\det(C_{w,z}) = \sigma_w^2 \sigma_z^2 - \rho^2 \sigma_w^2 \sigma_z^2$$

$$= (1-\rho^2)\sigma_w^2 \sigma_z^2$$

$$p_{w,z}(w,z) = \frac{1}{2\pi \det^{\frac{1}{2}}(C_{w,z})} e^{-\frac{1}{2}[w \ z] C_{w,z}^{-1} \begin{bmatrix} w \\ z \end{bmatrix}}$$

$$w \sim N(0, \sigma_w^2)$$

$$z \sim N(0, \sigma_z^2)$$

$$\begin{bmatrix} w \\ z \end{bmatrix} \sim N(\underset{2\times1}{\vec{0}}, \underset{2\times2}{C_{w,z}})$$

$$\begin{bmatrix} w \\ z \end{bmatrix} = G \begin{bmatrix} x \\ y \end{bmatrix}$$

$$C_{wz} = G C_{xy} G^T = \begin{bmatrix} \sigma_w & 0 \\ 0 & \sigma_z \end{bmatrix} \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \begin{bmatrix} \sigma_w & 0 \\ 0 & \sigma_z \end{bmatrix}$$

$$\begin{bmatrix} \sigma_w & \rho\sigma_z \\ \rho\sigma_w & \sigma_z \end{bmatrix}$$

eg. Affine Transformation of Standard Bivariate Gaussian

Non-linear

$$\begin{bmatrix} W \\ Z \end{bmatrix} = \underset{C_{w,z}}{\begin{bmatrix} \sigma_w & 0 \\ 0 & \sigma_z \end{bmatrix}} \begin{bmatrix} X \\ Y \end{bmatrix} + \underset{\substack{\uparrow \\ \text{Mean}}}{\begin{bmatrix} \mu_w \\ \mu_z \end{bmatrix}}$$

HW

$\mathbb{R}^2$

$$p_{W,Z}(w,z) = \frac{1}{2\pi \det^{\frac{1}{2}} C_{w,z}} e^{-\frac{1}{2}[w-\mu_w \;\; z-\mu_z] C_{w,z}^{-1} \begin{bmatrix} w-\mu_w \\ z-\mu_z \end{bmatrix}}$$

$$\begin{bmatrix} W \\ Z \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_w \\ \mu_z \end{bmatrix}, C_{w,z} \right)$$

eg.

$$\left.\begin{matrix} X \sim N(0,1) \\ Y \sim N(0,1) \end{matrix}\right] \text{Uncorrelated}$$

$$p_{X,Y}(x,y) = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}$$

$-\infty < x < \infty$

$-\infty < y < \infty$

$\rho = 0$

$X, Y \rightarrow$ Independent

$x = w, \; y = wz$

Non-linear $Z = \dfrac{Y}{X}$, $W = X$ $\qquad J^{-1}$

$$p_{W,Z}(w,z) \xrightarrow{\text{Marginalize}} p_Z(z) = \frac{1}{\pi} \frac{1}{(1+z^2)}$$

Cauchy PDF

$-\infty < z < \infty$

$$p_{w,z}(\omega,z) = p_{x,y}\left(g^{-1}(\omega,z),\ h^{-1}(\omega,z)\right) \left| \det \underbrace{\frac{\partial(x,y)}{\partial(\omega,z)}}_{J^{-1}} \right|$$

Jacobian $J^{-1} = \dfrac{\partial(x,y)}{\partial(\omega,z)} = \begin{bmatrix} \dfrac{\partial x}{\partial \omega} & \dfrac{\partial x}{\partial z} \\ \dfrac{\partial y}{\partial \omega} & \dfrac{\partial y}{\partial z} \end{bmatrix}$

Inverse transforms

$x = \omega, \quad y = \omega z$

$$J^{-1} = \begin{bmatrix} 1 & 0 \\ z & \omega \end{bmatrix} \qquad \det(J^{-1}) = \omega$$

Joint PDF

$$p_{w,z}(\omega,z) = \frac{1}{2\pi} e^{-\frac{1}{2}(\omega^2 + \omega^2 z^2)} \quad |\omega| \qquad S_{wz}$$

$$p_z(z) = \int_{\omega=-\infty}^{+\infty} \frac{1}{2\pi} e^{-\frac{1}{2}\omega^2(1+z^2)} \quad |\omega| \, d\omega$$

$$= \frac{\cancel{2}}{\cancel{2}\pi} \int_0^\infty \omega \, e^{-\frac{1}{2}\omega^2(1+z^2)} \, d\omega$$

$$= \frac{1}{2\pi} \int_0^\infty e^{-\frac{1}{2}\omega^2(1+z^2)} \, d(\omega^2)$$

$$= \frac{1}{2\pi} \left. \frac{e^{-\frac{1}{2}\omega^2(1+z^2)}}{-\frac{1}{2}(1+z^2)} \right|_0^\infty = \frac{1}{\pi} \cdot \frac{1}{1+z^2}$$

$-\infty < z < \infty$

Expected Values

$$E_{x,y}[xy] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy \; p_{x,y}(x,y)\, dx\, dy$$

Joint Expectation

$$z = g(x,y)$$

$$E_{x,y}[g(x,y)] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x,y)\; p_{x,y}(x,y)\, dx\, dy$$

If $z = g(x)$

$$E_{x,y}[g(x)] = \int_{x=-\infty}^{+\infty}\int_{y=-\infty}^{+\infty} g(x)\; p_{x,y}(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{+\infty} g(x)\; p_x(x)\, dx$$

$$= E_x[g(x)]$$

$$\mathrm{Cov}(x,y) = E_{x,y}\left[(x - E_x[x])\,(y - E_y[y])\right]$$

$$= E_{x,y}[xy] - E_x[x]\, E_y[y]$$

Similar to Discrete r.v.

$$\rho_{X,Y} = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{var}(X)}\sqrt{\mathrm{var}(Y)}}$$ Correlation Coefficient

$$-1 \leq \rho_{X,Y} \leq 1$$

Jointly Gaussian X, Y $\begin{bmatrix} X \\ Y \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix}, C_{X,Y}\right)$

$$p_{X,Y}(x,y) = \frac{1}{2\pi \det^{\frac{1}{2}}(C_{X,Y})} e^{-\frac{1}{2}\begin{bmatrix} x-\mu_X & y-\mu_Y \end{bmatrix} C_{X,Y}^{-1} \begin{bmatrix} x-\mu_X \\ y-\mu_Y \end{bmatrix}}$$

$C_{X,Y} \rightarrow$ Diagonal $\Rightarrow X,Y \rightarrow$ Uncorrelated

$\Rightarrow X,Y \rightarrow$ Independent

$$p_{X,Y} = p_X \, p_Y$$

Linear Transformation of Jointly Gaussian PDF is another Jointly Gaussian PDF.

Only the mean vector $\vec{\mu}$ and the Covariance matrix $C_{X,Y}$ are changed.

a) $X, Y \to$ zero mean, jointly Gaussian $X, Y \sim N(\vec{0}, C_{X,Y})$

Linear $\begin{bmatrix} W \\ Z \end{bmatrix} = G \begin{bmatrix} X \\ Y \end{bmatrix}$ $G \to$ Invertible

$$p_{W,Z}(w,z) = \frac{1}{2\pi \det^{\frac{1}{2}}(GC_{X,Y}G^T)} e^{-\frac{1}{2}[w \; z] (GC_{X,Y}G^T)^{-1} \begin{bmatrix} w \\ z \end{bmatrix}}$$

$\underline{HW}$

$C_{W,Z} = GC_{X,Y}G^T$ $\mu_W, \mu_Z = 0$, $\begin{bmatrix} W \\ Z \end{bmatrix} \sim N(\vec{0}, GC_{X,Y}G^T)$

b) General Affine $X, Y \to$ Jointly Gaussian, zero mean

$\begin{bmatrix} W \\ Z \end{bmatrix} = G \begin{bmatrix} X \\ Y \end{bmatrix} + \begin{bmatrix} \mu_W \\ \mu_Z \end{bmatrix}$ $G \to$ Invertible

$X, Y \sim N(\vec{0}, C_{X,Y})$

$$p_{W,Z}(w,z) = \frac{1}{2\pi \det^{\frac{1}{2}}(GC_{X,Y}G^T)} e^{-\frac{1}{2}[w-\mu_W \;\; z-\mu_Z] (GC_{X,Y}G^T)^{-1} \begin{bmatrix} w-\mu_W \\ z-\mu_Z \end{bmatrix}}$$

$\begin{bmatrix} W \\ Z \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_W \\ \mu_Z \end{bmatrix}, GC_{X,Y}G^T\right)$

c) $X, Y \sim N(\vec{\mu}, C_{X,Y})$

$\underline{HW}$

$\begin{bmatrix} W \\ Z \end{bmatrix} = G \begin{bmatrix} X \\ Y \end{bmatrix}$ $X, Y \to$ Jointly Gaussian, mean $\vec{\mu}$

$$\begin{bmatrix} W \\ Z \end{bmatrix} \sim N(G\vec{\mu}, GC_{X,Y}G^T)$$

Joint Moments

$$E_{X,Y}[g(X,Y)] = \int_x \int_y g(x,y)\, p_{X,Y}(x,y)\, dx\, dy$$

$$E_{X,Y}[X^k Y^l] = \iint_{x\ y} x^k y^l\, p_{X,Y}(x,y)\, dx\, dy$$

Central Moment

$$E_{X,Y}\left[(X-\mu_X)^k (Y-\mu_Y)^l\right] = \iint_{x\ y} (x-\mu_X)^k (y-\mu_Y)^l\, p_{X,Y}(x,y)\, dx\, dy$$

$X, Y \rightarrow$ Independent $\quad p_{X,Y}(x,y) = p_X(x)\, p_Y(y)$

$$\underset{X,Y}{E}[X^k Y^l] = E_X[X^k]\; E_Y[Y^l]$$

$$E_{X,Y}\left[(X-\mu_X)^k (Y-\mu_Y)^l\right] = E_X\left[(X-\mu_X)^k\right]\; E_Y\left[(Y-\mu_Y)^l\right]$$

Joint Characteristic Function

2D ICTFT

$$p_{X,Y}(x,y) = \frac{1}{(2\pi)^2} \int_{\omega_x} \int_{\omega_y} \Phi_{X,Y}(\omega_x,\omega_y)\, e^{-j(\omega_x x + \omega_y y)}\, d\omega_x\, d\omega_y$$

2D CTFT

$$\omega_{x|y} \rightarrow -\omega_{x|y}$$

$$\Phi_{X,Y}(\omega_x,\omega_y) = E_{X,Y}\left[e^{j(\omega_x X + \omega_y Y)}\right]$$

$$= \iint_{x\ y} e^{j(\omega_x x + \omega_y y)}\, p_{X,Y}(x,y)\, dx\, dy$$

$$E_{x,y}[x^k y^l] = \frac{1}{j^{k+l}} \frac{\partial^{k+l} \Phi_{x,y}(\omega_x, \omega_y)}{\partial \omega_x^k \, \partial \omega_y^l}\Bigg|_{\substack{\omega_x=0, \\ \omega_y=0}}$$

If $x, y \to$ independent $\Phi_{x,y}(\omega_x, \omega_y) = \Phi_x(\omega_x)\, \Phi_y(\omega_y)$

eg: $Z = X+Y$ $x, y \to$ Independent Given $p_x(x)$ $p_y(y)$

$$p_z(z) = p_x(z) * p_y(z) \longleftarrow$$ Continuous Convolution of PDFs

$p_x \to T_1$ $p_y \to T_2$

$p_z \to T_1 + T_2$

$$\Phi_z(\omega) = \Phi_x(\omega)\, \Phi_y(\omega)$$

$p_z(z) = \mathcal{F}^{-1}(\Phi_z(\omega))$ ICTFT

Proof similar to discrete case.

$$p_z(z) = \int_{t=-\infty}^{+\infty} p_x(t)\, p_z(z-t)\, dt$$

where

$$\Phi_x(\omega) = \int_{-\infty}^{+\infty} p_x(x)\, e^{j\omega x}\, dx$$

$$\Phi_y(\omega) = \int_{-\infty}^{+\infty} p_y(y)\, e^{j\omega y}\, dy$$

$$p_z(z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \Phi_z(\omega)\, e^{-j\omega z}\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \Phi_x(\omega)\, \Phi_y(\omega)\, e^{-j\omega z}\, d\omega$$

eg.

$X \sim N(\mu_x, \sigma_x^2)$ X, Y → Independent

$Y \sim N(\mu_y, \sigma_y^2)$

$Z = X+Y$

To show $Z \sim N(\mu_x+\mu_y, \sigma_x^2+\sigma_y^2)$

Method 1

$W = X$

$Z = X+Y$

$$\begin{bmatrix} W \\ Z \end{bmatrix} = \overset{G}{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}} \begin{bmatrix} X \\ Y \end{bmatrix}$$

$$p_{W,Z}(w,z) = p_{X,Y}\left(G^{-1}\begin{bmatrix} w \\ z \end{bmatrix}\right) \left|\det G^{-1}\right|$$ HW

$$p_Z(z) = \int_{w=-\infty}^{+\infty} p_{W,Z}(w,z)\,dw$$

Method 2

$$p_Z(z) = p_X(\underset{z}{\underset{\sim}{x}}) * p_Y(\underset{z}{\underline{y}}) = \int_{-\infty}^{+\infty} p_X(z')\, p_Y(z-z')\, dz'$$ HW

$$\Phi_Z(\omega) = \Phi_X(\omega)\, \Phi_Y(\omega)$$

$$\Phi_X(\omega) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\,\sigma_x} e^{-\frac{1}{2\sigma_x^2}(x-\mu_x)^2} e^{j\omega x}\, dx$$

$$\Phi_x(\omega) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\,\sigma_x} e^{-\frac{1}{2\sigma_x^2}(x^2 - 2\mu_x x + \mu_x^2) + j\omega x} dx$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\,\sigma_x} e^{-\frac{1}{2\sigma_x^2}[x^2 - 2\mu_x x + \mu_x^2 - j2\sigma_x^2 \omega x]} dx$$

$$= \cdot \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\,\sigma_x} e^{-\frac{1}{2\sigma_x^2}[x^2 - 2\mu_x x + \mu_x^2 - j2\sigma_x^2 \omega x]} \cdot e^{j\omega\mu_x} \boxed{e^{-j\omega\mu_x}} \cdot \boxed{e^{\frac{1}{2}\sigma_x^2\omega^2}} e^{-\frac{1}{2}\sigma_x^2\omega^2} dx$$

$$= e^{j\omega\mu_x - \frac{1}{2}\sigma_x^2\omega^2} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\,\sigma_x} e^{-\frac{1}{2\sigma_x^2}[x^2 - 2\mu_x x + 2j\omega\mu_x\sigma_x^2 + \mu_x^2 - j2\sigma_x^2\omega x - \sigma_x^4\omega^2]}$$

$$= e^{j\omega\mu_x - \frac{1}{2}\sigma_x^2\omega^2} \underbrace{\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\,\sigma_x} e^{-\frac{1}{2\sigma_x^2}[\overset{a^2}{x^2} - \overset{2a}{2x}\overset{b}{(\mu_x + j\sigma_x^2\omega)} + \underset{b^2}{(\mu_x + j\sigma_x^2\omega)^2}]} dx}_{\doteq 1 \qquad N(\mu_x + j\sigma_x^2\omega,\, \sigma_x^2)}$$

$$\Phi_x(\omega) = e^{j\omega\mu_x - \frac{1}{2}\sigma_x^2\omega^2}$$

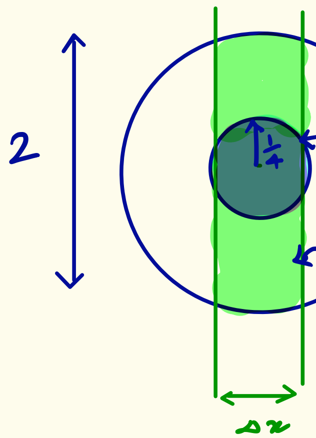
$$\Phi_y(\omega) = e^{j\omega\mu_y - \frac{1}{2}\sigma_y^2\omega^2}$$

$$\Phi_z(\omega) = e^{j\omega(\mu_x + \mu_y) - \frac{1}{2}\omega^2(\sigma_x^2 + \sigma_y^2)} \Rightarrow z \sim N(\mu_x + \mu_y,\, \sigma_x^2 + \sigma_y^2)$$

Conditional Probability Density Functions

eg. Dart board

$$p_{X,Y}(x,y) = \frac{1}{\pi}$$



$$P\left[\text{Bullseye} \,/\, \text{arrow hits } \Delta x \text{ width strip}\right]$$

$$= \frac{P[\qquad]}{P[\qquad]}$$

$$= \frac{\frac{1}{\pi}\,\Delta x \cdot \frac{1}{2}}{\frac{1}{\pi}\,\Delta x \cdot 2} = \frac{1}{4}$$

Conditional PDF

$$p_{X/Y}(x/y)$$

$\swarrow p_{X,Y}(x,y)$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{P\left[x-\frac{\Delta x}{2} \le X \le x+\frac{\Delta x}{2},\ y-\frac{\Delta y}{2} \le Y \le y+\frac{\Delta y}{2}\right]}{P\left[y-\frac{\Delta y}{2} \le Y \le y+\frac{\Delta y}{2}\right]} \nwarrow p_Y(y)$$

$p_{X,Y}(x,y)$ $\swarrow$

$$p_{Y/X}(y/x) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{P\left[x-\frac{\Delta x}{2} \le X \le x+\frac{\Delta x}{2},\ y-\frac{\Delta y}{2} \le Y \le y+\frac{\Delta y}{2}\right]}{P\left[x-\frac{\Delta x}{2} \le X \le x+\frac{\Delta x}{2}\right]} \nwarrow p_X(x)$$

$p_{Y/X}$, $p_{X/Y}$ → Family of uncountably infinite PDFs.

X, Y → Uncountably Infinite values

$$p_{Y/X}(y/x) = \frac{p_{X,Y}(x,y)}{p_X(x)} \longrightarrow \text{Family is function of } X$$

$$p_{X/Y}(x/y) = \frac{p_{X,Y}(x,y)}{p_Y(y)} \longrightarrow \text{Family is function of } Y$$

a) Conditional PDF from Joint PDF

$$p_{Y/X}(y/x) = \frac{p_{X,Y}(x,y)}{\int_{-\infty}^{+\infty} p_{X,Y}(x,y)\,dy} \qquad p_{X/Y}(x/y) = \frac{p_{X,Y}(x,y)}{\int_{-\infty}^{+\infty} p_{X,Y}(x,y)\,dx}$$

b) $$p_{X/Y}(x/y) = \frac{p_{Y/X}(y/x)\; p_X(x)}{p_Y(y)}$$

c) Bayes Rule $$p_{Y/X}(y/x) = \frac{p_{X/Y}(x/y)\; p_Y(y)}{\int_{-\infty}^{+\infty} p_{X/Y}(x/y)\; p_Y(y)\,dy}$$

d) $$p_{X,Y}(x,y) = p_{Y/X}(y/x)\; p_X(x) = p_{X/Y}(x/y)\; p_Y(y)$$

e) $$p_Y(y) = \int_{-\infty}^{+\infty} p_{Y/X}(y/x)\; p_X(x)\,dx$$

eg.

Standard Bivariate Gaussian PDF $p_{Y/X} = \frac{p_{X,Y}}{p_X}$

$X \sim N(0,1)$ $Y \sim N(0,1)$

$$p_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}(x^2-2\rho xy+y^2)}$$

$-\infty<x<\infty$

$-\infty<y<\infty$

$$p_{Y/X}(y/x) = \frac{\frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}(x^2-2\rho xy+y^2)}}{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}}$$

$p_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$

$$= \frac{1}{\sqrt{2\pi(1-\rho^2)}} e^{-\frac{1}{2(1-\rho^2)}\left[x^2-2\rho xy+y^2-(1-\rho^2)x^2\right]}$$

$$\cancel{x^2} \underline{-2\rho xy} + \underline{y^2} - \cancel{x^2} + \underline{\rho^2 x^2}$$

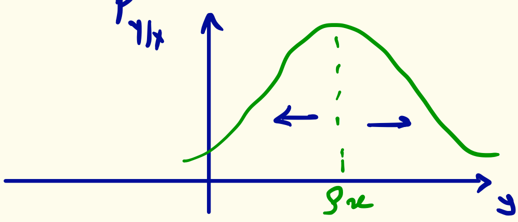
$$= \frac{1}{\sqrt{2\pi(1-\rho^2)}} e^{-\frac{1}{2(1-\rho^2)}(y-\rho x)^2}$$

→ function

$p_{Y/X} \sim N(\rho x, 1-\rho^2)$ of x

$$p_{X/Y}(x/y) = \frac{1}{\sqrt{2\pi(1-\rho^2)}} e^{-\frac{1}{2(1-\rho^2)}(x-\rho y)^2}$$

$$p_{X/Y} \sim N(\rho y, 1-\rho^2)$$



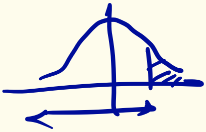
Conditional CDF

$$P[Y \le y / X = x] = \int_{t=-\infty}^{y} p_{Y/X}(t/x)\,dt = F_{Y/X}(y/x)$$

$$F_{X/Y}(x/y) = \int_{t=-\infty}^{x} p_{X/Y}(t/y)\,dt$$

eg.

$$Y/_{X=x} \sim N(\rho x, 1-\rho^2)$$



$$F_{Y/X}(y/x) = 1 - Q\left(\frac{y-\rho x}{\sqrt{1-\rho^2}}\right)$$

X, Y → Independent

$$p_{X,Y}(x,y) = p_X(x)\cdot p_Y(y)$$

$$p_{Y/X}(y/x) = \frac{p_X(x)\cdot p_Y(y)}{p_X(x)} = p_Y(y)$$

$$p_{X/Y}(x/y) = p_X(x)$$

$$F_{Y/X}(y/x) = F_Y(y)$$

$$F_{X/Y}(x/y) = F_X(x)$$

Application

X, Y → r. vs. → Independent

$Z = g(X, Y)$

$p_Z(z) = ?$

1. Fix $X = x$ $\quad Z/_{X=x} = g(x, Y)$

2. Transform from Y to Z using single random variable transformation.

$$p_{Z/X=x}$$

$$p_{Z/X=x}(z/x) = p_Y\left(g^{-1}(z,x)\right)\left|\frac{\partial g^{-1}(z,x)}{\partial z}\right|$$

3. Uncondition

$$p_Z(z) = \int_{x=-\infty}^{+\infty} p_{Z,X}(z,x)\,dx$$

$$= \int_{x=-\infty}^{+\infty} p_{Z/X}(z/x)\,p_X(x)\,dx$$

Expectation of a Conditional PDF

$$E_{Y/X}\left[Y/X\right]_{X=x} = \int_{-\infty}^{+\infty} y \; p_{Y/X}(y/x)\, dy$$

In general $E_{Y/X}\left[Y/X\right] \rightarrow$ Function of X.

$$E_{X/Y}\left[X/Y\right]_{Y=y} = \int_{-\infty}^{+\infty} x \; p_{X/Y}(x/y)\, dx$$

$E_{X/Y}\left[X/Y\right] \rightarrow$ Function of Y.

Unconditioning

$$E_Y\left[E_{X/Y}[X/Y]\right] = E_X[X]$$

$$E_X\left[E_{Y/X}[Y/X]\right] = E_Y[Y]$$

$X/_{Y=y} \sim N(\rho y, 1-\rho^2)$

$E_{X/Y}[X/Y] = \rho Y$

function of Y

eg. Bivariate Gaussian

$$Y/_{X=x} \sim N(\rho x, 1-\rho^2)$$

$E_{Y/X}[Y/X] = \rho X \rightarrow$ function of X

Continuous N-D Random Variables (Random vectors)

$$\vec{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} \in \mathbb{R}^N$$

point in $\mathbb{R}^N$

No. of possible values $\vec{X}$ → Infinite

$$S_{X_1, X_2, \ldots, X_N} \subseteq \mathbb{R}^N$$

eg: Temperature Profile for N successive days.

Joint PDF

$$p_{X_1, X_2, \ldots, X_N}(x_1, x_2, \ldots, x_N) = p_{\vec{X}}(\vec{x}) \geq 0$$

N-D hyper volume

$$\int_{x_1}\int_{x_2} \cdots \int_{x_N} p_{\vec{X}}(\vec{x})\, dx_1 \ldots dx_N = 1$$

eg. Multivariate Gaussian PDF $\vec{X} \sim N(\vec{\mu}_{\vec{X}}, C_{\vec{X}})$

$$P_{\vec{X}}(\vec{x}) = \frac{1}{(2\pi)^{N/2} \det^{\frac{1}{2}}(C_{\vec{X}})} e^{-\frac{1}{2}(\vec{x}-\vec{\mu}_{\vec{X}})^T C_{\vec{X}}^{-1}(\vec{x}-\vec{\mu}_{\vec{X}})}$$

$$\vec{\mu}_{\vec{X}} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_N \end{bmatrix}_{N\times 1} = E_{\vec{X}}[\vec{X}] = E_{\vec{X}}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} E_{x_1}[x_1] \\ E_{x_2}[x_2] \\ \vdots \\ E_{x_N}[x_N] \end{bmatrix}$$

$$E_{x_1}[x_1] = \mu_1, \quad \cdots, \quad E_{x_N}[x_N] = \mu_N$$

$C_{\vec{X}} \rightarrow N \times N$ Covariance Matrix

$$C_{\vec{X}} = \begin{bmatrix} \mathrm{var}(x_1) & \mathrm{Cov}(x_1,x_2) & \cdots & \mathrm{Cov}(x_1,x_N) \\ \mathrm{Cov}(x_1,x_2) & \mathrm{var}(x_2) & & \vdots \\ \vdots & & \ddots & \\ \mathrm{Cov}(x_1,x_N) & & \cdots & \mathrm{Var}(x_N) \end{bmatrix}$$

$$= E_{\vec{X}}\left[(\vec{X}-\vec{\mu})(\vec{X}-\vec{\mu})^T\right]$$

Marginal PDFs

$$p_{X_1}(x_1) = \int_{x_2} \cdots \int_{x_N} p_{\vec{X}}(\vec{x}) \, dx_2 \ldots dx_N$$

eg. Bivariate from Multivariate Gaussian.

eg. Multivariate Gaussian

$$\vec{X} \sim N\left(\vec{\mu}_{\vec{X}}, C_{\vec{X}}\right) \quad p_{\vec{X}}(\vec{x}) = \frac{1}{(2\pi)^{N/2} \det^{\frac{1}{2}}(C_{\vec{X}})} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})^T C_{\vec{X}}^{-1} (\vec{x}-\vec{\mu})}$$

$$x_1, x_2, \ldots, x_N \rightarrow \text{Uncorrelated.} \Rightarrow C_{\vec{X}} = \begin{bmatrix} \sigma_1^2 & & 0 \\ & \sigma_2^2 & \\ & & \ddots \\ 0 & & & \sigma_N^2 \end{bmatrix}$$

$$C_{\vec{X}}^{-1}$$

$$p_{\vec{X}}(\vec{x}) = \frac{1}{(2\pi)^{N/2} \det^{\frac{1}{2}}(C_{\vec{X}})} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})^T \begin{bmatrix} \frac{1}{\sigma_1^2} & & 0 \\ & \frac{1}{\sigma_2^2} & \\ 0 & & \ddots \; \frac{1}{\sigma_N^2} \end{bmatrix} (\vec{x}-\vec{\mu})}$$

$$= \frac{1}{\prod_{i=1}^{N} \sqrt{2\pi} \, \det^{\frac{1}{2}}(C_{\vec{X}})} e^{-\frac{1}{2}[x_1-\mu_1 \ \ldots \ x_N-\mu_N] \begin{bmatrix} \frac{1}{\sigma_1^2} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\sigma_N^2} \end{bmatrix} \begin{bmatrix} x_1-\mu_1 \\ \vdots \\ x_N-\mu_N \end{bmatrix}}$$

$$= \frac{1}{\prod_{i=1}^{N} \sqrt{2\pi} \left(\prod_{i=1}^{N} \sigma_i^2 \right)^{\frac{1}{2}}} e^{-\frac{1}{2} \sum_{i=1}^{N} \frac{(x_i-\mu_i)^2}{\sigma_i^2}}$$

$$= \frac{1}{\prod_{i=1}^{N} \sqrt{2\pi \sigma_i^2}} \prod_{i=1}^{N} e^{-\frac{1}{2} \frac{(x_i-\mu_i)^2}{\sigma_i^2}}$$

Joint PDF

$$p_{\vec{X}}(\vec{x}) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}\,\sigma_i} e^{-\frac{1}{2\sigma_i^2}(x_i-\mu_i)^2}$$

$$= \prod_{i=1}^{N} p_{X_i}(x_i) \quad \text{N Marginal PDFs}$$

$$p_{X_i}(x_i) = \frac{1}{\sqrt{2\pi}\,\sigma_i} e^{-\frac{1}{2\sigma_i^2}(x_i-\mu_i)^2} \quad , \quad X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$$

$\Rightarrow$ $X_1, X_2, \dots, X_N$ are independent.

$\Rightarrow$ All Higher order moments can also be factorized.

$$E_{X_1,X_2,\dots,X_N}\left[X_1^{l_1} X_2^{l_2} \cdots X_N^{l_N}\right] = E_{X_1}\left[X_1^{l_1}\right] \cdots E_{X_N}\left[X_N^{l_N}\right]$$

Joint CDF

$$F_{X_1,X_2,\dots,X_N}(x_1,x_2,\dots,x_N) = \int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\cdots\int_{-\infty}^{x_N} p_{\vec{X}}(\vec{k})\, dk_1 \dots dk_N \qquad \vec{k} = \begin{bmatrix} k_1 \\ \vdots \\ k_N \end{bmatrix}$$

$$F_{X_1,X_2,\dots,X_N}(-\infty,-\infty,\dots,-\infty) = 0$$

$$F_{X_1,X_2,\dots,X_N}(\infty,\infty,\dots,\infty) = 1$$

Marginal CDF

$$F_{X_i}(x_i) = F_{X_1, X_2, \ldots, X_N}(\infty, \infty, \ldots, x_i, \infty, \ldots \infty)$$

Transformation of Random Vector (continuous)

General Transformation

$$\vec{X} \text{ to } \vec{Y} \qquad \vec{Y} = \vec{g}(\vec{X})$$

$$Y_1 = g_1(X_1, \ldots, X_N) \qquad X_1 = g_1^{-1}(Y_1, \ldots, Y_N)$$

$$Y_2 = g_2(X_1, \ldots, X_N) \qquad X_2 = g_2^{-1}(Y_1, \ldots, Y_N)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$Y_N = g_N(X_1, \ldots, X_N) \qquad X_N = g_N^{-1}(Y_1, \ldots, Y_N)$$

$$p_{Y_1, Y_2, \ldots, Y_N}(y_1, y_2, \ldots, y_N)$$

$$= p_{X_1, X_2, \ldots, X_N}(g_1^{-1}(\vec{y}), g_2^{-1}(\vec{y}), \ldots, g_N^{-1}(\vec{y})) \left| \det\left(\frac{\partial(x_1, x_2, \ldots, x_N)}{\partial(y_1, y_2, \ldots, y_N)} \right) \right|$$

$$\frac{\partial(x_1,\dots,x_N)}{\partial(y_1,\dots,y_N)} = \begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_N} \\ \vdots & & & \\ \frac{\partial x_N}{\partial y_1} & \frac{\partial x_N}{\partial y_2} & \cdots & \frac{\partial x_N}{\partial y_N} \end{bmatrix}$$

Linear Transformation

$$\underset{M\times 1}{\vec{Y}} = \underset{M\times N}{G}\ \underset{N\times 1}{\vec{X}} \qquad M<N$$

$\tilde{G}_{N\times N} \rightarrow (N-M)$ rows of the form $\quad \text{rank}(\tilde{G}_{N\times N}) = N$

$$0 \; 0 \; \cdots 0 \; 1 \; 0 \cdots 0$$

$$Y_{M+1} = X_{M+1}, \; \dots, \; Y_N = X_N$$

$$\vec{Y}_{N\times 1} = G_{N\times N}\,\vec{X}_{N\times 1}$$

$G \rightarrow$ Invertible

$$G = \begin{bmatrix} \vec{g}_1^{T} \\ \vec{g}_2^{T} \\ \vdots \\ \vec{g}_N^{T} \end{bmatrix}$$

$$p_{\vec{Y}}(\vec{y}) = p_{\vec{X}}\left(G^{-1}\vec{y}\right)\left|\det(G^{-1})\right|$$

eg. Multivariate Gaussian

$$\vec{X} \sim N(\mu_{\vec{X}}, C_{\vec{X}})$$

$$\underset{N\times 1}{\vec{Y}} = \underset{N\times N}{G}\ \underset{N\times 1}{\vec{X}} \qquad \text{rank}(G) = N$$

To show

$$\vec{Y} \sim N(G\mu_{\vec{X}}, G C_{\vec{X}} G^T)$$ also a Multivariate Gaussian

$$p_{\vec{Y}}(\vec{y}) = p_{\vec{X}}(G^{-1}\vec{y})\,|\det G^{-1}|$$

$$= \frac{1}{(2\pi)^{N/2}\det^{\frac{1}{2}}(C_{\vec{X}})} e^{-\frac{1}{2}(G^{-1}\vec{y} - \vec{\mu}_{\vec{X}})^T C_{\vec{X}}^{-1}(G^{-1}\vec{y} - \vec{\mu}_{\vec{X}})} \cdot \frac{1}{|\det G|}$$

$$\det^{\frac{1}{2}}(C_{\vec{X}})\,|\det G| = \det^{\frac{1}{2}}(C_{\vec{X}})\sqrt{\det(GG^T)} = \det^{\frac{1}{2}}(G C_{\vec{X}} G^T)$$

$$\left[\det(GG^T)\right]^{\frac{1}{2}} = \det G \qquad G^{-T} C_{\vec{X}}^{-1} G^{-1}$$

$$p_{\vec{Y}}(\vec{y}) = \frac{1}{(2\pi)^{N/2}\det^{\frac{1}{2}}(\underbrace{G C_{\vec{X}} G^T})} e^{-\frac{1}{2}(\vec{y} - \underbrace{G\vec{\mu}_{\vec{X}}}_{\vec{\mu}_{\vec{Y}}})^T \overbrace{(G C_{\vec{X}} G^T)}^{C_{\vec{Y}}}{}^{-1}(\vec{y} - G\vec{\mu}_{\vec{X}})}$$

$$C_{\vec{Y}} = G C_{\vec{X}} G^T \qquad \vec{\mu}_{\vec{Y}} = G\vec{\mu}_{\vec{X}}$$

Expected Values of Random Vector

$$E_{\underline{\vec{X}}}[\underline{\vec{X}}] = \begin{bmatrix} E_{X_1}[X_1] \\ E_{X_2}[X_2] \\ \vdots \\ E_{X_N}[X_N] \end{bmatrix}$$

$$E_{\underline{\vec{X}}}\left[g(x_1, x_2, \ldots, x_N)\right]$$

$$= \int_{x_1} \int_{x_2} \cdots \int_{x_N} g(x_1, x_2, \ldots, x_N) \; p_{\underline{\vec{X}}}(\vec{x}) \, dx_1 \ldots dx_N$$

$$\text{Var}\left(\sum_{i=1}^{N} a_i X_i\right) = \sum_{i=1}^{N} \sum_{j=1}^{N} a_i a_j \, \text{Cov}(X_i, X_j)$$

$$= \vec{a}^{\,T} C_{\underline{\vec{X}}} \, \vec{a} \geq 0$$

$$\sum_{i=1}^{N} a_i X_i = \vec{a}^{\,T} \underline{\vec{X}}$$

If $X_1, X_2, \ldots, X_N$ → Uncorrelated

$$\text{Var}\left(\sum_{i=1}^{N} a_i X_i\right) = \sum_{i=1}^{N} a_i^2 \, \text{Var}(X_i)$$

Independent Identically Distributed Random Variables

$x_1, x_2, \dots, x_N$ are IID Random Variables.

$p_{\vec{X}}(\vec{x}) = p_{x_1}(x_1) \dots p_{x_N}(x_N)$ eg. $x_i \sim N(0,{}^2)$

$x_i \rightarrow$ Mean μ

$x_i \rightarrow$ Variance σ^2

Sample Mean $$\hat{X} = \frac{1}{N} \sum_{i=1}^{N} x_i$$

$$= \begin{bmatrix} \frac{1}{N} & \frac{1}{N} & \cdots & \frac{1}{N} \end{bmatrix}_{1 \times N} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}_{N \times 1}$$

$$E_{\vec{X}}[\hat{X}] = \frac{1}{N} \sum_{i=1}^{N} E_{x_i}[x_i]$$

$$= \frac{1}{N} \sum_{i=1}^{N} \mu$$

$$= \frac{1}{N} \cdot N \mu$$

$$= \mu$$

$$\text{Var}(\hat{X}) = \text{Var}\left(\frac{1}{N} \sum_{i=1}^{N} x_i\right)$$

$$= \frac{1}{N^2} \sum_{i=1}^{N} \text{Var}(x_i)$$

x_i

$\mu \rightarrow$ Mean

$\sigma^2 \rightarrow$ Variance

$$\text{Var}(\hat{x}) = \frac{1}{N^2} \sum_{i=1}^{N} \sigma^2$$

$$= \frac{1}{N^2} N \sigma^2$$

$$= \frac{\sigma^2}{N}$$

As $N \to \infty$ $\quad \text{var}(\hat{x}) \longrightarrow 0$

$\hat{x} \longrightarrow$ Single peak at μ with zero width (prob 1)

Nate Silver — Signal and the Noise

Joint Moments

$$E_{x_1, x_2, \ldots, x_N}\left[x_1^{l_1} \ldots x_N^{l_N}\right] = \int_{x_1}\int_{x_2} \ldots \int_{x_N} x_1^{l_1} x_2^{l_2} \ldots x_N^{l_N} \, p_{\vec{X}}(\vec{x}) \, dx_1 \ldots dx_N$$

If $x_1, x_2, \ldots, x_N$ are independent

$$E_{x_1, x_2, \ldots, x_N}\left[x_1^{l_1} \ldots x_N^{l_N}\right] = E_{x_1}\left[x_1^{l_1}\right] \ldots E_{x_N}\left[x_N^{l_N}\right]$$

Joint Characteristic Function

$$\Phi_{X_1,X_2,\ldots,X_N}(\omega_1,\omega_2,\ldots,\omega_N) = E_{\vec{X}}\left[e^{j(\omega_1 X_1+\cdots+\omega_N X_N)}\right]$$

$$= \int_{x_1}\int_{x_2}\cdots\int_{x_N} e^{j(\omega_1 x_1+\cdots+\omega_N x_N)}\, P_{\vec{X}}(\vec{x})\, dx_1 \ldots dx_N$$

N-D CTFT

$$P_{\vec{X}}(\vec{x}) = \frac{1}{(2\pi)^N}\int_{\omega_1}\cdots\int_{\omega_N} \Phi_{X_1,X_2,\ldots,X_N}(\omega_1,\omega_2,\ldots,\omega_N)\, e^{-j(\omega_1 x_1+\cdots+\omega_N x_N)}\, d\omega_1 \ldots d\omega_N$$

N-D ICTFT

$$E_{X_1,X_2,\ldots,X_N}\left[X_1^{l_1} X_2^{l_2} \cdots X_N^{l_N}\right] = \frac{1}{j^{l_1+l_2+\cdots+l_N}} \frac{\partial^{l_1+l_2+\cdots+l_N}\,\Phi_{X_1,\ldots,X_N}(\omega_1,\ldots,\omega_N)}{\partial\omega_1^{l_1}\cdots\partial\omega_N^{l_N}}\Bigg|_{\omega_1,\omega_2,\ldots,\omega_N=0}$$

eg: IID Sum

$$\vec{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$$

$x_1, x_2, \ldots, x_N \longrightarrow$ IID

$$p_{\vec{X}}(\vec{x}) = p_{x_1}(x_1) \ldots p_{x_N}(x_N)$$

$$Y = \sum_{i=1}^{N} x_i = [1 \quad 1 \quad \ldots \quad 1] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$$

$x_i \sim U(0,2)$

$i = 1, 2, \ldots, N$

$$p_y(y) = p_{x_1}(x_1) * p_{x_2}(x_2) \ldots * p_{x_N}(x_N)$$

$$\Phi_y(\omega) = E_y\left[e^{j\omega y}\right]$$

$$= E_{x_1, x_2, \ldots, x_N}\left[e^{j\omega \sum_{i=1}^{N} x_i}\right]$$

$$E_{x_i}\left[e^{j\omega x_i}\right] = \Phi_x(\omega)$$

$$= E_{x_1, x_2, \ldots, x_N}\left[\prod_{i=1}^{N} e^{j\omega x_i}\right]$$

as $x_1, x_2, \ldots, x_N$ have same PDF $p_x(x)$

$$= \prod_{i=1}^{N} E_{x_i}\left(e^{j\omega x_i}\right)$$

$$= \prod_{i=1}^{N} \Phi_{x_i}(\omega)$$

$$\Phi_y(\omega) = \left[\Phi_x(\omega)\right]^N$$

$$p_y(y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[\Phi_x(\omega)\right]^N e^{-j\omega y} \, d\omega$$

$$p_y(y) = \underbrace{p_x(x) * p_x(x) * \ldots * p_x(x)}_{N\text{-1 times}}$$

$$\frac{1}{2}[1\ \ 1] * \frac{1}{2}[1\ \ 1] = \frac{1}{4}[1\ \ 2\ \ 1]$$

$$\frac{1}{4}[1\ 2\ 1] * \frac{1}{4}[1\ 2\ 1] = \frac{1}{16}[1\ \ 4\ \ 6\ \ 4\ \ 1]$$

$$\frac{1}{16}[1\ 4\ 6\ 4\ 1] * \frac{1}{16}[1\ 4\ 6\ 4\ 1] = \frac{1}{256}[1\ \ 8\ \ 28\ \ 56\ \ 70\ \ 56\ \ 28\ \ 8\ \ 1]$$

Linear Prediction MMSE

Case a)

X → Random Variable

Minimum Mean Square Error Estimation

$$E\left[(X-\hat{x})^2\right] \rightarrow \text{Minimized.}$$

Minimizer $\hat{x} = E[X]$

Error $\text{var}(X) = E\left[(X - E[X])^2\right]$

Case b)

X, Y → Random Variables

$$\hat{Y} = aX + b$$

$$E\left[(Y-\hat{Y})^2\right] \rightarrow \text{Minimized.}$$

Minimizer $\hat{Y} = E_Y[Y] + \frac{\text{Cov}(X,Y)}{\text{Var}(X)}\left(X - E_X[X]\right)$

$$\hat{Y} = \underbrace{\frac{\text{Cov}(X,Y)}{\text{Var}(X)}}_{a_{opt}} X \underbrace{- \frac{\text{Cov}(X,Y)}{\text{Var}(X)} E_X[X] + E_Y[Y]}_{b_{opt}}$$

Optimum when X, Y → Bivariate Gaussian.

Case c)

$$x_1, x_2, x_3, \ldots, x_p \rightarrow \text{Random Variables (ordered)}$$

$$\hat{x}_{p+1} = \sum_{i=1}^{p} a_i \, x_i \qquad C_{i,j} = Cov(x_i, x_j) = E_{x_i x_j}[x_i x_j] \quad i \neq j$$

$$x_1, x_2, \ldots, x_p \longrightarrow E_{x_i}[x_i] = 0 \quad \forall \; i = 1, 2, \ldots, p$$

$a_i \rightarrow$ Linear Prediction Coefficients.

Find a_i's which minimize $\quad \vec{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{p+1} \end{bmatrix}$

$$\underset{\vec{X}}{E}\left[(x_{p+1} - \hat{x}_{p+1})^2 \right] = \underset{\vec{X}}{E}\left[\left(x_{p+1} - \sum_{i=1}^{p} a_i x_i\right)^2 \right]$$

$$\frac{\partial}{\partial a_1} \underset{\vec{X}}{E}\left[\left(x_{p+1} - \sum_{i=1}^{p} a_i x_i\right)^2 \right] = 0$$

$$E_{\vec{X}}[x_1 x_{p+1}] - E_{\vec{X}}\left[\sum_{i=1}^{p} a_i x_1 x_i\right] = 0$$

$$E_{\vec{X}}\left[-2\left(x_{p+1} - \sum_{i=1}^{p} a_i x_i\right) x_1 \right] = 0$$

$$E_{x_1, x_{p+1}}[x_1 \, x_{p+1}] - \sum_{i=1}^{p} a_i \, E_{x_1, x_i}[x_1 x_i] = 0$$

$$C_{1,p+1} = \sum_{i=1}^{p} a_i \, C_{1,i}$$

In general $\frac{\partial}{\partial a_j} = 0 \quad j = 1, 2, \dots, p$

$$C_{j,p+1} = \sum_{i=1}^{p} a_i C_{j,i} \qquad j = 1, 2, \dots, p$$

C → Covariance as all x_i's have zero mean

$$\underset{p \times p}{\begin{bmatrix} C_{1,1} & C_{1,2} & \cdots & C_{1,p} \\ C_{2,1} & C_{2,2} & \cdots & C_{2,p} \\ \vdots & C_{3,3} & & \vdots \\ & & \ddots & \vdots \\ C_{p,1} & C_{p,2} & \cdots & C_{p,p} \end{bmatrix}} \underset{p \times 1}{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix}} = \underset{p \times 1}{\begin{bmatrix} C_{1,p+1} \\ C_{2,p+1} \\ \vdots \\ C_{p,p+1} \end{bmatrix}}$$

$$C_{\vec{\hat{X}}} \, \vec{a}_{opt} = \vec{c}$$

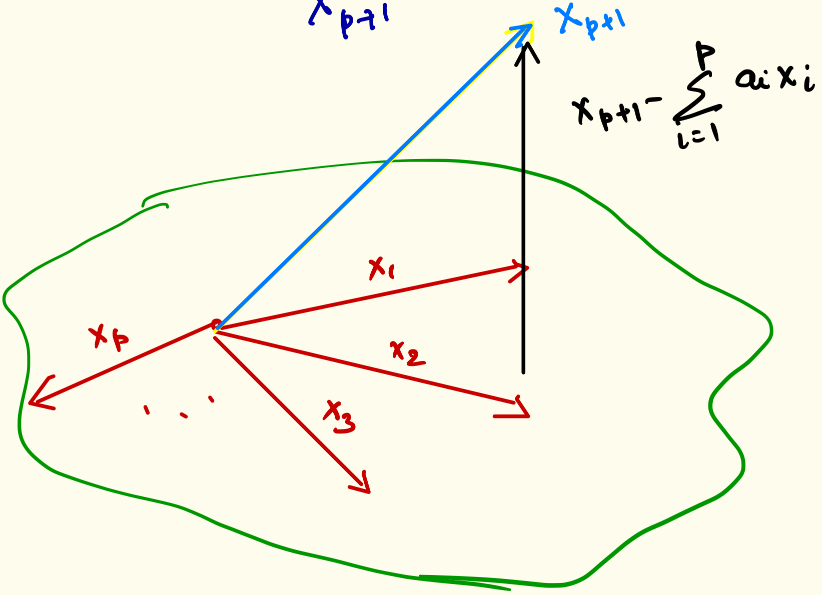
$$\vec{\hat{X}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}$$

$$\vec{a}_{opt} = C_{\vec{\hat{X}}}^{-1} \, \vec{c}$$

Orthogonality Principle

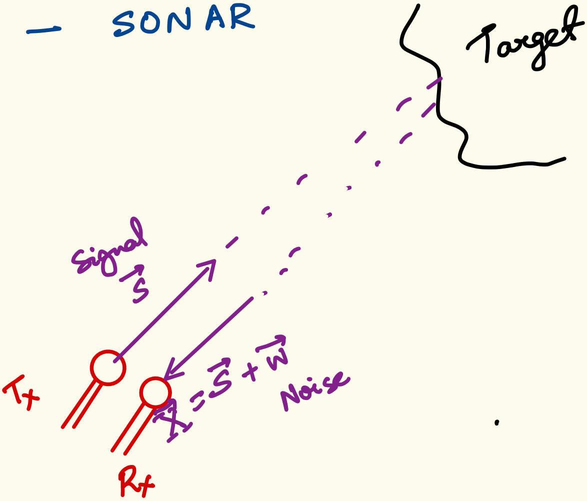
$$E_{\vec{X}}\left[-2\left(x_{p+1} - \sum_{i=1}^{p} a_i x_i\right) x_1\right] = 0$$

$$E_{\vec{X}}\left[\left(x_{p+1} - \underset{\hat{X}_{p+1}}{\sum_{i=1}^{p} a_i x_i}\right) x_1\right] = 0 \qquad a_1$$

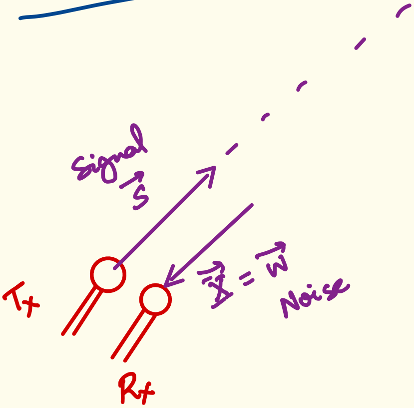


Signal Detection

- RADAR
- SONAR



No Target



Signal Detection

– RADAR / SONAR

Signal sent $\vec{S} = \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_N \end{bmatrix}$ → Deterministic

Signal Received $\vec{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$ → Random Vector

$$p_{W_i}(w_i) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{w_i^2}{\sigma^2}}$$

No Target $x_i = W_i$ $\quad W_i \sim N(0, \sigma^2)$

Target Present $x_i = s_i + W_i$ $\quad \vec{W} = \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_N \end{bmatrix}$ → Uncorrelated

$C_{\vec{W}}$ → diagonal

$$\det(C_{\vec{W}}) = (\sigma^2)^N, \quad C_{\vec{W}} = \begin{bmatrix} \sigma^2 & & 0 \\ & \sigma^2 \ddots & \\ 0 & & \sigma^2 \end{bmatrix} = \sigma^2 I_{N \times N}$$

Hypotheses

H_W : $\quad x_i = W_i \qquad i = 1, 2, \dots, N$ $\quad$ No Target.

H_{S+W} : $\quad x_i = s_i + W_i \qquad i = 1, 2, \dots, N$ $\quad$ Target Present.

No Target $P_{\vec{X}}(\vec{x}; H_W)$ → PDF when only noise present

Target Present $P_{\vec{X}}(\vec{x}; H_{S+W})$ → PDF when both signal and noise are present.

Target Detected if $p_{\vec{X}}(\vec{x}; H_{s+w}) > p_{\vec{X}}(\vec{x}; H_w)$

White

$$H_w: \vec{X} = \vec{W} \sim N(\vec{0}, \sigma^2 I_{N\times N}) \rightarrow p_{\vec{X}}(\vec{x}; H_w)$$

Affine

$$H_{s+w}: \vec{X} = \vec{s} + \vec{W} \sim N(\vec{s}, \sigma^2 I_{N\times N}) \rightarrow p_{\vec{X}}(\vec{x}; H_{s+w})$$

$$\frac{1}{(2\pi)^{N} \det^{\frac{1}{2}}(C_{\vec{X}})} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})^T C_{\vec{X}}^{-1}(\vec{x}-\vec{\mu})}$$

Target Detected if

$$\cancel{\frac{1}{(2\pi\sigma^2)^{N/2}}} e^{-\frac{1}{2\sigma^2}(\vec{x}-\vec{s})^T(\vec{x}-\vec{s})} > \cancel{\frac{1}{(2\pi\sigma^2)^{N/2}}} e^{-\frac{1}{2\sigma^2}\vec{x}^T\vec{x}}$$

$$\ln e^{-\frac{1}{2\sigma^2}(\vec{x}-\vec{s})^T(\vec{x}-\vec{s})} > \ln e^{-\frac{1}{2\sigma^2}\vec{x}^T\vec{x}}$$

$$-\left[(\vec{x}-\vec{s})^T(\vec{x}-\vec{s})\right] > -\vec{x}^T\vec{x}$$

$$\cancel{-\vec{x}^T\vec{x}} + 2\vec{x}^T\vec{s} - \vec{s}^T\vec{s} > \cancel{-\vec{x}^T\vec{x}} \Rightarrow 2\vec{x}^T\vec{s} - \vec{s}^T\vec{s} > 0$$

$$\vec{x}^T\vec{s} > \frac{1}{2}\vec{s}^T\vec{s}$$

$\vec{s}$ → signal sent

$\vec{x}$ → signal receiv

$$\sum_{i=1}^{N} x_i s_i > \frac{1}{2}\sum_{i=1}^{N} s_i^2$$

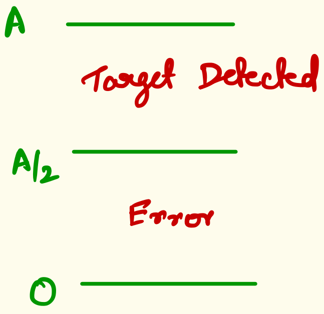
$$\vec{s} = \begin{bmatrix} A \\ A \\ \vdots \\ A \end{bmatrix}_{N\times 1} \rightarrow \text{DC signal.}$$

$$A \sum_{i=1}^{N} x_i > \frac{1}{2} N A^2$$

Target Detected

Decision Rule

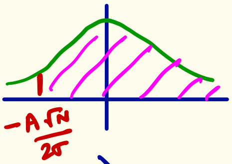
$$\sum_{i=1}^{N} x_i > \frac{1}{2} N A$$



$$\frac{1}{N} \sum_{i=1}^{N} x_i > \frac{A}{2}$$

$$\bar{X}_N > A/2$$

$$\bar{X}_N \sim \mathcal{N}(A, \sigma^2/N)$$



$$P[\bar{X}_N > A/2]$$

$$= P\Big[\underbrace{\frac{1}{N} \sum_{i=1}^{N} x_i}_{\bar{X}_N} > A/2\Big] = Q\left(\frac{A/2 - A}{\sqrt{\sigma^2/N}}\right)$$

$$= Q\left(\frac{-A/2}{\sqrt{\sigma^2/N}}\right)$$

$$= Q\left(-\frac{A\sqrt{N}}{2\sigma}\right)$$

$P[\bar{X}_N > A/2] \longrightarrow 1$ as $N \to \infty$ Infinite length

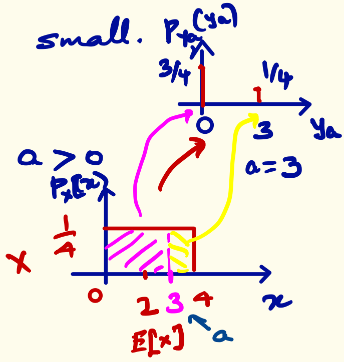
or as $A \to \infty$ Power

Markov Inequality

$X \rightarrow$ Random Variable takes only Non-negative Values.

$\hookrightarrow \mathbb{Z}^+$ or $\mathbb{R}^+$

If X has small mean, then probability of X taking a large value must be small.



$$P[X \geq a] \leq \frac{E[X]}{a} \quad , \quad \forall\, a > 0$$

Proof

For fixed a, define

$$Y_a = \begin{cases} 0 & , \text{ if } X < a \\ a & , \text{ if } X \geq a \end{cases}$$

$$\Rightarrow \quad Y_a \leq X$$

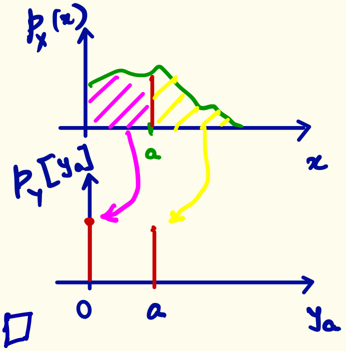
$$\Rightarrow \quad E[Y_a] \leq E[X] \quad \text{———} \; ①$$

$$E[Y_a] = a\, P[Y_a = a] = a\, P[X \geq a] \quad \text{———} \; ②$$

From ① + ②

$$a\, P[X \geq a] \leq E[X]$$

$$\Rightarrow \quad P[X \geq a] \leq \frac{E[X]}{a}$$

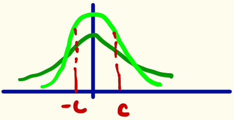


□

Chebyshev Inequality

$X \rightarrow$ Random Variable with mean μ, variance σ^2

If a random variable has small variance, then the probability that it takes values far from its mean is also small.



$$P[|X-\mu| \geq c] \leq \frac{\sigma^2}{c^2}, \quad \forall \; c>0$$

Proof

Let $a = c^2$

$Y = (X-\mu)^2 \longrightarrow$ takes only non-negative values

By Markov Inequality

$$P[Y \geq a] \leq \frac{E_Y[Y]}{a}$$

$$\Rightarrow \quad P[(x-\mu)^2 \geq c^2] \leq \frac{E_x[(x-\mu)^2]}{c^2} \rightarrow \sigma^2$$

as $c>0$, $(x-\mu)^2 \geq c^2 \Rightarrow |x-\mu| \geq c$

$$P[|x-\mu| \geq c] \leq \frac{\sigma^2}{c^2}$$

Let $c = k\sigma$, $k>0$

$$P[|x-\mu| \geq k\sigma] \leq \frac{1}{k^2}$$

eg. Exponential PDF

$X \sim exp(\lambda)$

Let $\lambda = 2$

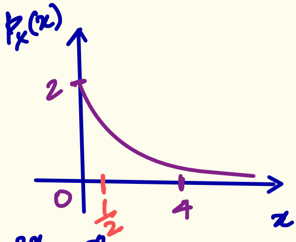
$X \sim exp(2)$

$$p_X(x) = \begin{cases} 2e^{-2x}, & x \geq 0 \\ 0, & o/w \end{cases}$$

$$\int_0^\infty 2e^{-2x} dx = 2 \frac{e^{-2x}}{-2} \Big|_0^\infty$$

$$= 2\left[0 - \left(-\frac{1}{2}\right)\right]$$

$$= 1$$



$E_X[X] = \frac{1}{2}$, $var(X) = \frac{1}{4}$

Actual

$$P[X \geq 4] = \int_4^\infty 2e^{-2x} dx = 2 \frac{e^{-2x}}{-2} \Big|_4^\infty = 2\left[0 + \frac{e^{-8}}{2}\right]$$

$$= e^{-8} \approx 0.0003$$

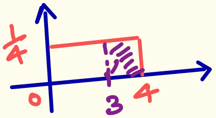
Markov

$$P[X \geq 4] \leq \frac{\frac{1}{2}}{4} \text{ or } \frac{1}{8} \qquad P[X \geq 10] \leq \frac{1}{20}$$

Chebyshev

$$P\left[\left|X - \frac{1}{2}\right| \geq 4\right] \leq \frac{\frac{1}{4}}{16} \text{ or } \frac{1}{64}$$

$$P\left[\left|X - \frac{1}{2}\right| \geq 10\right] \leq \frac{1}{400}$$



eg. $X \sim U(0, 4)$ $E_X[X] = 2$, $var(X) = \frac{16}{12} = \frac{4}{3}$

Markov $P[X \geq 3] \leq \frac{2}{3}$

Actual $P[X \geq 3] = \frac{1}{4}$

eg. $X \sim U(-4, +4)$ $E_X[X] = 0$ $var(X) = \frac{64}{12}$

Chebyshev $P[\,|X| \geq 3] \leq \frac{\frac{64}{12}}{9} = \frac{64}{108} = \frac{16}{27}$

Actual $P[|X| \geq 3] = \frac{1}{4}$

Chernoff Inequality

$X \rightarrow$ Random Variable

Let $M(s) = E[e^{sX}]$ Moment Generating Function
$s \in \mathbb{R}$

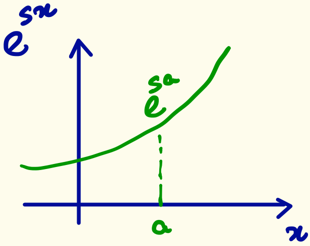
$$P[X \geq a] \leq e^{-sa} M(s) \quad \forall a \quad \forall s \geq 0$$

$$P[X \leq a] \leq e^{-sa} M(s) \quad \forall a \quad \forall s < 0$$

Proof

a) Given some a, $s \geq 0$,

$$Y_a = \begin{cases} 0, & \text{if } X < a \\ e^{sa}, & \text{if } X \geq a \end{cases}$$



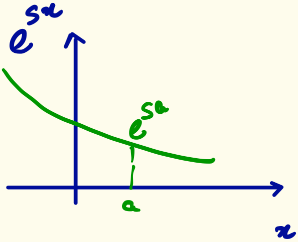
$$Y_a \leq e^{sX} \Rightarrow E[Y_a] \leq \underset{M(s)}{E[e^{sX}]} \quad \text{—— ①}$$

$$E[Y_a] = e^{sa} P[Y_a = e^{sa}] = e^{sa} \cdot P[X \geq a] \quad \text{—— ②}$$

$$\Rightarrow \quad e^{sa} P[X \geq a] \leq M(s)$$

$$P[X \geq a] \leq e^{-sa} M(s)$$

b) Given some a, $s < 0$,



$$Y_a = \begin{cases} e^{sa} & , \text{ if } x \leq a \\ 0 & , \text{ if } x > a \end{cases}$$

$$Y_a \leq e^{sX} \Rightarrow E[Y_a] \leq \underset{M(s)}{E\left[e^{sX}\right]} \quad — ①$$

$$E[Y_a] = e^{sa} P\left[Y_a = e^{sa}\right] = e^{sa} P[X \leq a] \quad — ②$$

$$\Rightarrow P[X \leq a] \leq e^{-sa} M(s)$$

Moment Generating Function

PDF Cont.

$$M(s) = E\left[e^{sX}\right] = \int_{-\infty}^{+\infty} e^{sx} p_X(x)\, dx$$

PMF Disc.

$$M(s) = E\left[e^{sX}\right] = \sum_{-\infty}^{+\infty} e^{sk} p_X[k]$$

Law of Large Numbers

$x_1, x_2, \ldots, x_N$ are IID random variables with mean $E_x[x]$ and $\text{var}(x) = \sigma^2 < \infty$

then $$\lim_{N\to\infty} \bar{x}_N = E_x[x] \quad , \quad \bar{x}_N = \frac{1}{N}\sum_{i=1}^{N} x_i$$

Proof

To prove that the probability of sample mean r.v. $\bar{x}_N$ deviating from expected value by more than ϵ is zero.

$$\lim_{N\to\infty} P\left[\left|\bar{x}_N - E_x[x]\right| > \epsilon\right] = 0$$

$\epsilon \to$ small +ve number

$$P\left[\left|\bar{x}_N - E_x[x]\right| > \epsilon\right] \leq \frac{\text{var}(\bar{x}_N)}{\epsilon^2}$$

By chebyshev

$$\leq \frac{\sigma^2}{N\epsilon^2}$$

As $N \to \infty$

$$\lim_{N\to\infty} P\left[\left|\bar{x}_N - E_x[x]\right| > \epsilon\right] \leq 0$$

Prob. cannot be < 0

$$\Rightarrow \lim_{N\to\infty} P\left[\left|\bar{x}_N - E_x[x]\right| > \epsilon\right] = 0$$

eg: Signal Detection

$$X_{s_i+w_i} \sim N(A, \sigma^2) \qquad i=1,2,\dots,N$$

R.Vs are IID

$$\overline{X}_N = \frac{1}{N}\sum_{i=1}^{N} X_{s_i+w_i} \xrightarrow[\text{LLN}]{\text{as } N\to\infty} \overline{X}_N = A$$

$$\overline{X}_N > A/2$$

$$X_{s_i+w_i} \sim U(0, 2A) \qquad i=1,2,\dots,N$$

R.Vs are IID

$$\overline{X}_N \xrightarrow[\text{LLN}]{N\to\infty} \overline{X}_N = A$$

Central Limit Theorem

$$X_i \sim U\left(-\tfrac{1}{2}, \tfrac{1}{2}\right) \qquad \text{IID}$$

$$Y = X_1 + X_2 + \dots + X_N$$

$$p_Y(y) = \underbrace{p_X(x) * p_X(x) * \dots * p_X(x)}_{N-1 \text{ convolutions}}$$

$$p_x(x) * p_x(x) \longrightarrow \mathcal{N}(0, 2/12)$$

Conv
MATLAB

$$p_x(x) * p_x(x) * p_x(x) \longrightarrow \mathcal{N}(0, 3/12)$$

$$\underbrace{p_x(x) * p_x(x) * \dots * p_x(x)}_{N-1 \text{ conv.}} \longrightarrow \mathcal{N}(0, N/12)$$

$$S_N = X_1 + X_2 + \dots + X_N \qquad X_i \rightarrow \text{IID}$$

$$E_{X_i}[X_i] = E_X[X]$$

$$\text{var}(X_i) = \text{var}(X)$$

Normalized / standardized Sum

IID

$$E[S_N] = N\, E_X[X]$$

$$\text{var}(S_N) = N\, \text{var}(X)$$

$$\tilde{S}_N = \frac{S_N - E[S_N]}{\sqrt{\text{var}(S_N)}} = \frac{S_N - N\, E_x[x]}{\sqrt{N\, \text{var}(x)}}$$

As $N \rightarrow \infty$

$$\tilde{S}_N \longrightarrow \mathcal{N}(0, 1)$$

CLT

PDF of a standardized sum of a large number of continuous IID r.v.s will converge to a Gaussian PDF $\mathcal{N}(0,1)$.

Applications

- Polling Prediction
- Noise Modeling
- scattering effects modeling
- kinetic theory of gases
- Economics / stock market.

CLT

$x_1, x_2, \ldots, x_N \rightarrow$ continuous IID r.vs.

each with mean $E_x[x]$ and variance $\text{var}(x)$.

$$S_N = \sum_{i=1}^{N} x_i$$

As $N \rightarrow \infty$

$$\tilde{S}_N = \frac{S_N - E[S_N]}{\sqrt{\text{var}(S_N)}} = \frac{\sum_{i=1}^{N} x_i - N E_x[x]}{\sqrt{N \,\text{var}(x)}} \longrightarrow \mathcal{N}(0,1)$$

$$P\left[\frac{S_N - E[S_N]}{\sqrt{\text{var}(S_N)}} \leq x\right] \longrightarrow \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} dt$$

eg. $\{X_i\}_{i=1}^{N} \longrightarrow$ Bernoulli IID r.vs.

$$S_N = \sum_{i=1}^{N} x_i$$

Binomial $p_{S_N}[k] = \binom{N}{k} p^k (1-p)^{N-k}$ $\quad k = 0, 1, \dots, N$

$E[S_N] = Np$

As $N \to \infty$

$$\tilde{S}_N = \frac{S_N - E[S_N]}{\sqrt{\mathrm{var}(S_N)}} = \frac{S_N - Np}{\sqrt{Np(1-p)}} \longrightarrow \mathcal{N}(0,1)$$

By CLT Samples

Random Processes

eg. Bernoulli Trials

- Infinite number of coin Tosses.
- Infinite number of Coins.

H → 1

T → 0

$n \rightarrow$

$\{ \; (0, 1, 1, 1, \ldots),$

$s \downarrow \; (1, 0, 1, 1, \ldots),$

$(1, 1, 0, 1, \ldots), \rightarrow$ Sample sequence

$\vdots$

$\}$

n → n^{th} coin toss

s → s^{th} coin

Ensemble

$X(n, s)$

n → Time Index

s → sample Index

Toss 1st 2nd 3rd

Sample sequences

Coin1 $(1, 0, 0, 1, 0, 1, 1, 1, 0, \dots)$ $x_1[n]$ $X[n, s_1]$

Coin2 $(1, 1, 0, 1, 1, 0, 1, 1, 1, \dots)$ $x_2[n]$ $X[n, s_2]$

Coin3 $(0, 0, 1, 1, 0, 0, 1, 0, 0, \dots)$ $x_3[n]$ $X[n, s_3]$

$\vdots$ $\vdots$

$X[5, s] \equiv X[5]$

Random Variable

Random Process $X[n]$

Coin

1st $x_1[n]$

2nd $x_2[n]$

$\vdots$

$\longrightarrow$ Sample sequences or Realizations or Sample Functions $\Big\}$ Ensemble of Realizations

Toss

1st $X[1]$

2nd $X[2]$

3rd $X[3]$

$\vdots$

$\longrightarrow$ Random variables (Bernoulli)

Random Process $X[n]$

$$X[n] \rightarrow \begin{cases} x_1[n] \\ x_2[n] \\ \vdots \\ x_\infty[n] \end{cases}$$

∞

Sample sequences

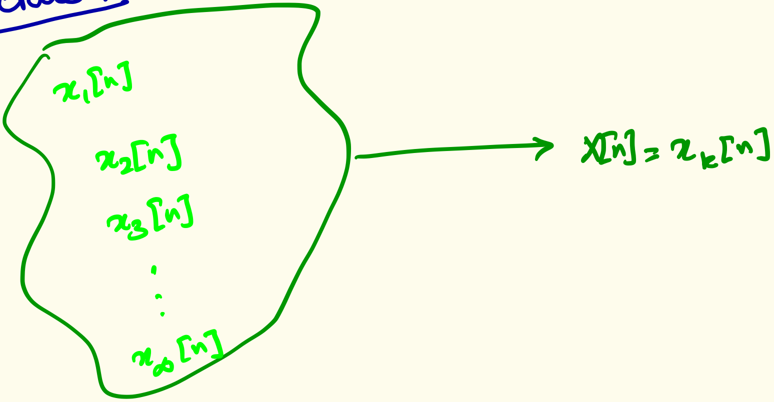
or

Realizations

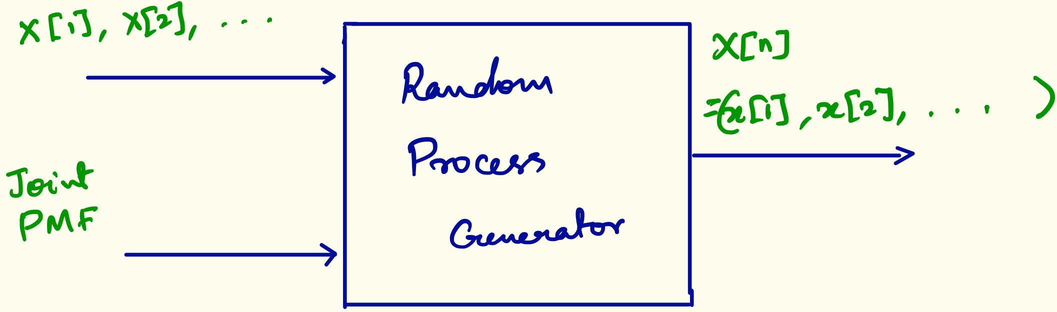
∞ Random Variables

$X[n]$ $\quad X[1] \quad X[2] \quad . \; . \; . \quad X[\infty]$

Picture 1



Picture 2



$$\begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$$

Random vector

Realization

eg. Bernoulli Random Process

$$P[\text{Heads in all first 5 coin tosses}]$$

$$P[x[1]=1,\ x[2]=1,\ x[3]=1,\ x[4]=1,\ x[5]=1]$$

$$= P\left[\begin{bmatrix} x[1] \\ \vdots \\ x[5] \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}\right]$$

$$= \prod_{i=1}^{5} P[x[i]=1] = p^5$$

Classification of Random Processes

a) Infinite $X[n]$, $-\infty < n < \infty$

b) Semi Infinite $x[n]$, $0 \le n < \infty$

$x[n]$, $-\infty < n \le 0$

Finite → Random Vector

X[n] Sample sequences

a) Discrete Time Discrete Valued (DTDV)
n PMF eg. Bernoulli Process

b) Discrete Time Continuous Valued (DTCV)
n PDF eg. Gaussian r.v at every discrete time

X(t) Sample functions

c) Continuous Time Discrete Valued (CTDV)
t PMF

d) Continuous Time Continuous Valued (CTCV)
t PDF

Realization

$X[n] \rightarrow$ Discrete Time Sample sequence $x_k[n]$

$X(t) \rightarrow$ Continuous Time Sample Function $x_k(t)$

eg. Random Walk

$$U[i] = \begin{cases} -1 \\ +1 \end{cases} \qquad p_U[k] = \begin{cases} \frac{1}{2}, & k=-1 \\ \frac{1}{2}, & k=+1 \end{cases}$$

$$X[n] = \sum_{i=0}^{n} U[i] \quad , \quad U[i] \rightarrow \text{IID RP}$$

$$p_U[k] = \begin{cases} \frac{1}{2}, k=-1 \\ \frac{1}{2}, k=+1 \end{cases}$$

$$E\left[U[i]\right] = +1 \cdot \frac{1}{2} - 1 \cdot \frac{1}{2} = 0$$

$$\operatorname{var}\left(U[i]\right) = 1^2 \cdot \frac{1}{2} + (-1)^2 \cdot \frac{1}{2} = 1$$

$$X[n] = \sum_{i=0}^{n} U[i]$$

$$E\left[X[n]\right] = (n+1) \cdot 0 = 0$$

$$\operatorname{var}\left(X[n]\right) = \sum_{i=0}^{n} \operatorname{var}\left(U[i]\right)$$

$$= (n+1) \cdot \operatorname{var}\left(U[0]\right)$$

$$= (n+1) \cdot 1$$

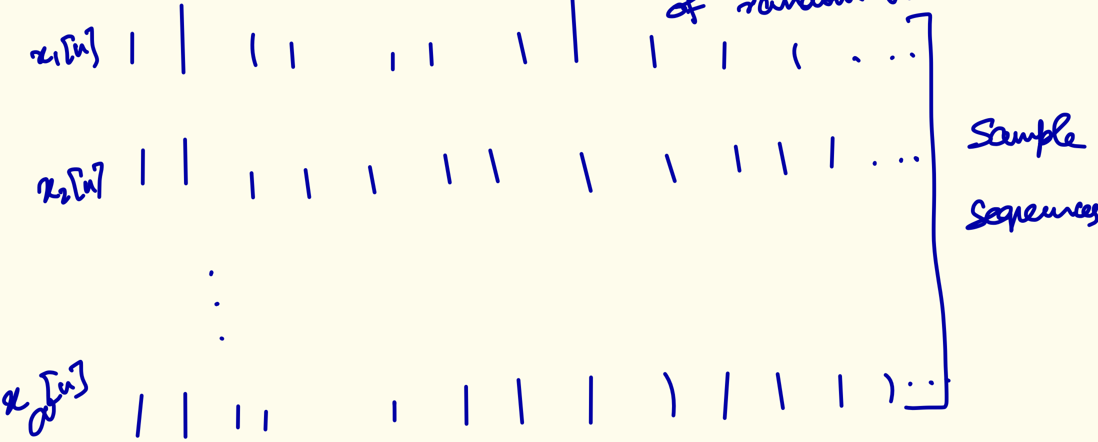
$$= n+1$$

As $n \to \infty$, CLT $\quad X[n] = \lim_{n\to\infty} \sum_{i=0}^{n} U[i]$

$$X[n] \sim \mathcal{N}(0, n+1)$$

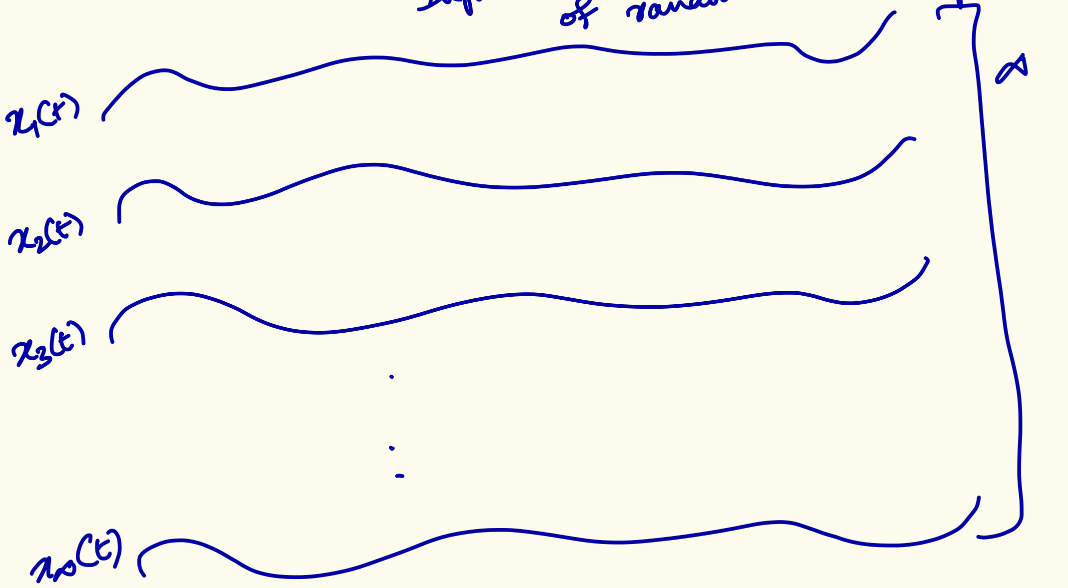
$$\tilde{X}[n] = \frac{X[n] - E\left[X[n]\right]}{\sqrt{\operatorname{var}\left(X[n]\right)}} = \frac{X[n]}{\sqrt{n+1}} \sim \mathcal{N}(0,1)$$

DT RP $X[n]$ Countably Infinite number of random variables

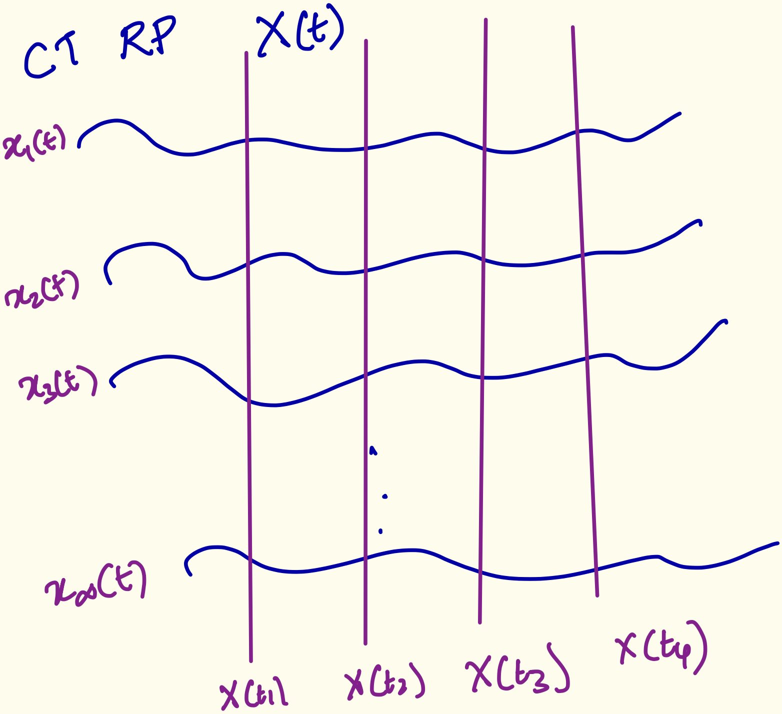


CT RP $X(t)$ Uncountably Infinite number of random variables

Sample functions



CT RP $X(t)$



$$p_{X(t_1),X(t_2),X(t_3),\, X(t_4)} \rightarrow \text{Joint PMF/PDF}$$

FDD

Finite Dimensional Distribution (FDD)

Joint PDF/PMF of any finite set of random variables sampled from a random process.

Joint PDF/PMF of the random vector obtained from a random process.

IID Random Process (Independent Identically Distributed)

PDF/PMF of samples of a DT random process $X[n]$

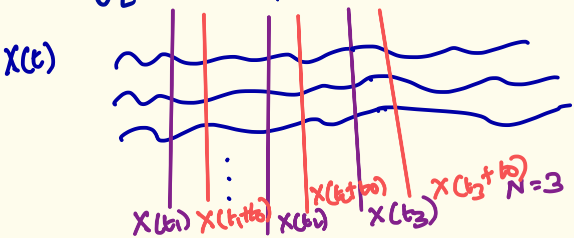
N locations $\{ X[n_1], X[n_2], \ldots, X[n_N] \} \rightarrow N$ r.v.s.

$X[n_i] \rightarrow$ same PDF/PMF

$X[n_i]$ & $X[n_j] \rightarrow$ Independent if $i \neq j$

$$p_{X[n_i], X[n_j]} = p_{X[n_i]} \, p_{X[n_j]}$$

eg. $V[i]$ in previous examples



Stationarity

DT Random Process $X[n]$ is strict sense stationary if FDD does not change with time origin.

$$p_{X[n_1+n_0],\, X[n_2+n_0],\, \ldots,\, X[n_N+n_0]} = p_{X[n_1],\, X[n_2],\, \ldots,\, X[n_N]}$$

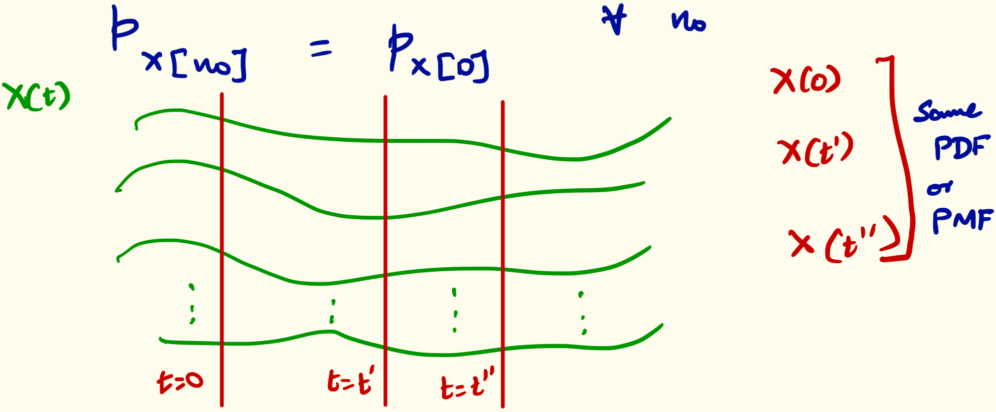
for all n_0, for any arbitrary N, and for any $n_1, n_2, \ldots, n_N$

if $N=1$

$$p_{X[n_1+n_0]} = p_{X[n_1]}$$

let $n_1=0$

$$p_{X[n_0]} = p_{X[0]} \quad \forall \; n_0$$



Any IID Random Process is also strict sense stationary.

<u>Proof</u> FDD

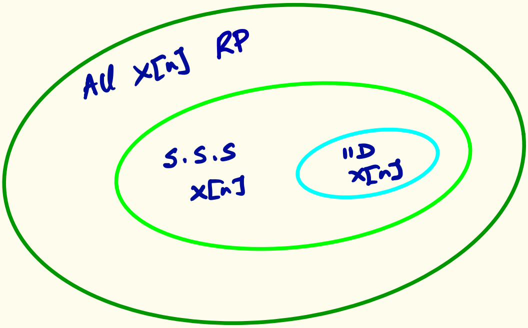
$$p_{x[n_1+n_0],\ x[n_2+n_0],\ \ldots,\ x[n_N+n_0]}$$

$$= \prod_{i=1}^{N} p_{x[n_i+n_0]}$$ Independence

$$= \prod_{i=1}^{N} p_{x[n_i]}$$ Identically Distributed

$$= p_{x[n_1],\ x[n_2],\ \ldots,\ x[n_N]}$$ Independence

$\Rightarrow$ $x[n]$ is strict sense stationary.



If $x[n]$ is stationary, all joint moments are also stationary.

$$E_{x[n_1+n_0],\, x[n_2+n_0],\, \dots,\, x[n_N+n_0]}[\cdot] = E_{x[n_1],\, \dots,\, x[n_N]}[\cdot]$$

Examples of Random Processes

a) Sum Random Process DTCV | DTDV

$$X[n] = \sum_{i=0}^{n} U[i] \quad \text{—①}, \quad U[i] \text{ are IID RP}$$

↖ same PDF | PMF

$$E[X[n]] = (n+1)\, E[U[0]]$$

$$\text{var}(X[n]) = (n+1)\, \text{var}(U[0])$$

E, var → functions of n

$\Rightarrow$ $X[n]$ is non-stationary.

$$X[n-1] = \sum_{i=0}^{n-1} U[i] \quad \text{—②}$$

① - ②

$$U[n] = X[n] - X[n-1], \quad X[-1] = 0$$

$U[i] \rightarrow$ IID $\qquad X[n_2] = \sum_{i=0}^{n_2} U[i] \qquad X[n_1] = \sum_{i=0}^{n_1} U[i]$

$$X[n_2] - X[n_1] = \sum_{i=n_1+1}^{n_2} U[i]$$

$$X[n_4] - X[n_3] = \sum_{i=n_3+1}^{n_4} U[i]$$

Independent increment

$n_4 > n_3 \geq n_2 > n_1 \qquad$ Non-overlap

$n_4 - n_3 = n_2 - n_1 \qquad \left.\begin{array}{l} X[n_4] - X[n_3] \\ X[n_2] - X[n_1] \end{array}\right] \rightarrow$ Identical number of $U[i]$

$\Rightarrow$ Convolution of same number of PDFs/PMFs ($U[i]$)

Arrange all independent increments to create a Random Process which IID $\Rightarrow$ Stationary.

b) Binomial Counting Random Process DTDV

$U[n] \longrightarrow$ Bernoulli RP IID

$$p_U[k] = \begin{cases} p, & k=1 \\ 1-p, & k=0 \end{cases}$$

$$U[n] = \begin{cases} 1, & \text{with prob. } p \\ 0, & \text{with prob. } (1-p) \end{cases}$$

Binomial Counting

$$X[n] = \sum_{i=0}^{n} U[i] \qquad n = 0, 1, 2, \dots$$

$$X[n] = \begin{cases} U[0], & n = 0 \\ X[n-1] + U[n], & n \geq 1 \end{cases}$$

$U[n] \rightarrow$ stationary

eg:

$$P\left[X[1] = 1, \; X[2] = 2\right]$$

$$= P\left[U[0] + U[1] = 1, \; U[2] = 1\right]$$

$$= P\left[U[0] + U[1] = 1\right] P\left[U[2] = 1\right]$$

$$= \binom{2}{1} p(1-p) \cdot p$$

$$= 2p^2(1-p)$$

$X[n] \rightarrow$ Non-stationary.

c) White Gaussian Noise Process (WGN)

IID process with marginal PDF

DTCV ✓
CTCV

$X[n] \sim \mathcal{N}(0, \sigma^2) \quad -\infty < n < \infty$

each r.v. $X[n_0]$ → zero mean, variance σ^2

$p_{X[n_0]}(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}$

WGN → Stationary as it is an IID process.

FDD

Joint PDF

$$p_{X[n_1], X[n_2], \dots, X[n_N]}(x_1, x_2, \dots, x_N) = \prod_{i=1}^{N} p_{X[n_i]}(x_i)$$ Independe

$$= \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{x_i^2}{\sigma^2}}$$

$$= \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{N} x_i^2}$$

$$= \frac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2}\vec{x}^T C^{-1} \vec{x}}$$

$$C = \begin{bmatrix} \sigma^2 & & 0 \\ & \sigma^2 & \\ 0 & & \ddots \; \sigma^2 \end{bmatrix}_{N\times N} \qquad \sim \mathcal{N}(\vec{0}, \sigma^2 I_{N\times N})$$

d) Moving Average Random Process

DTCV

$U[n] \rightarrow$ WGN RP with $\mathcal{N}(0, \sigma_U^2)$
IID

Stationary $X[n] = \frac{1}{2}\left(U[n] + U[n-1]\right) \Rightarrow p_{X[n]} = \frac{1}{2}\, p_{U[n]} * p_{U[n-1]}$

Joint PDF of $X[0], X[1]$ FDD

$$\underset{\vec{X}}{\begin{bmatrix} X[0] \\ X[1] \end{bmatrix}} = \underset{G}{\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}} \underset{\vec{U}}{\begin{bmatrix} U[-1] \\ U[0] \\ U[1] \end{bmatrix}}$$

$$\vec{X} = G\,\vec{U}$$

$$\vec{X} \sim \mathcal{N}\left(G\,E[\vec{U}],\ G\,C_U\,G^T\right)$$

$$\vec{X} \sim \mathcal{N}\left(\vec{0},\ \sigma^2\,G\,G^T\right)$$

$$G\,G^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

Joint Moments

countably infinite random variables

For DT Random Process $x[n]$ $\quad n \in \mathbb{Z}$

Mean Sequence $\quad \mu_x[n] = E\left[X[n]\right] \quad -\infty < n < \infty$

Variance sequence $\quad \sigma_x^2[n] = \mathrm{var}\left(X[n]\right) \quad -\infty < n < \infty$

Covariance Sequence $\quad$ 2D infinite matrix in both dimensio

$$C_x[n_1, n_2] = \mathrm{Cov}\left(X[n_1], X[n_2]\right) \qquad \begin{matrix} -\infty < n_1 < \infty \\ -\infty < n_2 < \infty \end{matrix}$$

$$= E\left[\left(X[n_1] - \mu_x[n_1]\right)\left(X[n_2] - \mu_x[n_2]\right)\right]$$

$$= E\left[X[n_1]\, X[n_2]\right] - E\left[X[n_1]\right] E\left[X[n_2]\right]$$

$$= E\left[X[n_1]\, X[n_2]\right] - \mu_x[n_1]\, \mu_x[n_2]$$

$$C_x[n_1, n_2] = C_x[n_2, n_1]$$

$$C_x[n, n] = \sigma_x^2[n] = \mathrm{var}\left(X[n]\right)$$

For a CT Random Process $X(t)$ Uncountably infinite random variables $t \in \mathbb{R}$

Mean function $\mu_X(t) = E[X(t)]$ $-\infty < t < \infty$

Variance function $\sigma_X^2(t) = \text{var}(X(t))$ $-\infty < t < \infty$

Covariance function $C_X(t_1, t_2)$ 2D surface infinite in t_1, t_2

$-\infty < t_1 < \infty$

$= E\left[\left(X(t_1) - \mu_X(t_1)\right)\left(X(t_2) - \mu_X(t_2)\right)\right]$ $-\infty < t_2 < \infty$

$= E[X(t_1) X(t_2)] - \mu_X(t_1)\mu_X(t_2)$

eg. WGN → White Gaussian Noise RP

$X[n] \sim \mathcal{N}(0, \sigma^2) \quad \forall n \in \mathbb{Z}$ IID RP

$\mu_X[n] = 0, \quad -\infty < n < \infty$

Kronecker Delta

$$\delta[n_1 - n_2] = \begin{cases} 1, & n_1 = n_2 \\ 0, & n_1 \neq n_2 \end{cases}$$

$\sigma_X^2[n] = \sigma^2, \quad -\infty < n < \infty$

$$C_X[n_1, n_2] = \begin{cases} 0, & n_1 \neq n_2 \\ \sigma^2, & n_1 = n_2 \end{cases} = \sigma^2 \delta[n_1 - n_2]$$

eg: Moving Average RP

$$X[n] = \frac{1}{2}\left(U[n] + U[n-1]\right)$$

$-\infty < n < \infty$

$U[n] \sim \mathcal{N}(0, \sigma_U^2)$

$U[n] \rightarrow$ IID

$$E[X[n]] = \frac{1}{2}\left(E[U[n]] + E[U[n-1]]\right)$$

$E[U[n]] = 0$

$\text{Var}(U[n]) = \sigma_U^2$

$$\mu_X[n] = 0 \quad \forall n$$

$$C_X[n_1, n_2] = E\left[X[n_1]\, X[n_2]\right]$$

$$= \frac{1}{4} E\left[\left(U[n_1] + U[n_1-1]\right)\left(U[n_2] + U[n_2-1]\right)\right]$$

$$= \frac{1}{4}\Big[E[U[n_1]\, U[n_2]] + E[U[n_1-1]\, U[n_2-1]] + E[U[n_1]\, U[n_2-1]] + E[U[n_1-1]\, U[n_2]] \Big]$$

$$E[U[k]\, U[l]] = \begin{cases} 0, & \text{if } k \neq l \\ \sigma_U^2, & \text{if } k = l \end{cases}$$

$$= \sigma_U^2\, \delta[k-l] = \sigma_U^2\, \delta[l-k]$$

$$C_X[n_1, n_2] = \frac{1}{4}\Big[2\sigma_U^2\, \delta[n_2 - n_1] + \sigma_U^2 \delta[n_2 - n_1 - 1] + \sigma_U^2 \delta[n_2 - n_1 + 1] \Big]$$

$$C_x[n_1, n_2] = \begin{cases} \frac{1}{2}\sigma_v^2, & n_1 = n_2 \\ \frac{1}{4}\sigma_v^2, & |n_2 - n_1| = 1 \\ 0, & |n_2 - n_1| > 1 \end{cases}$$

$$C_x[n_1, n_2] = \frac{1}{2}\sigma_v^2 \delta[n_2 - n_1] + \frac{1}{4}\sigma_v^2 \delta[n_2 - n_1 - 1] + \frac{1}{4}\sigma_v^2 \delta[n_2 - n_1 + 1]$$

Wide Sense Stationary Random Processes

WSS

A D R.P $x[n]$ is WSS if

a) $\mu_x[n] = \mu$ (constant) $-\infty < n < \infty$

b) $C_x[n_1, n_2] = g(|n_2 - n_1|)$ $-\infty < n_1, n_2 < \infty$

$\Rightarrow$ $E[x[n]] = \mu$, $-\infty < n < \infty$

$$E[x[n_1]\, x[n_2]] = C_x[n_1, n_2] + \mu^2$$

$$= h(|n_2 - n_1|) \quad -\infty < n_1, n_2 < \infty$$

eg. Moving Average R.P.

If $x[n]$ is a SSS R P, then it is also a WSS R P.

Proof If $x[n]$ is a SSS RP

$$p_{x[n+n_0]} = p_{x[n]} \quad \forall\, n, n_0$$

Let $n=0$

$$p_{x[n_0]} = p_{x[0]} \quad \forall\, n_0$$

$$\Longrightarrow \quad \mu_x[n] = \mu \qquad -\infty < n < \infty$$

with $N=2$

$$p_{x[n_1+n_0],\, x[n_2+n_0]} = p_{x[n_1],\, x[n_2]} \quad \forall\, n_1, n_2, n_0$$

Let $n_0 = -n_1$

$$p_{x[0],\, x[n_2-n_1]} = p_{x[n_1],\, x[n_2]} \quad \text{——} ①$$

Let $n_0 = -n_2$

$$p_{x[0],\, x[n_1-n_2]} = p_{x[n_1],\, x[n_2]} \quad \text{——} ②$$

$$p_{x[0],\,x[n_2-n_1]} = p_{x[0],\,x[n_1-n_2]} = p_{x[n_1],\,x[n_2]}$$

$$C_x[n_1,n_2] + \mu^2$$

$$= E\left[x[n_1]\,x[n_2]\right] = E\left[x[0]\,x[n_2-n_1]\right]$$

$$= E\left[x[0]\,x[n_1-n_2]\right]$$

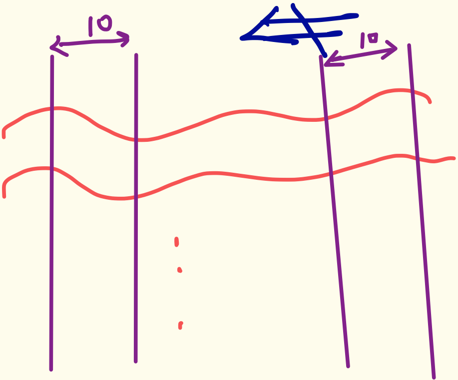
$$= h(|n_2-n_1|)$$

$\Rightarrow$ $x[n]$ is WSS RP.

WSS RP relies only on conditions on first order (μ_x) and second order (C_x) moments of the RP.

$x[n]$ is SSS $\Longrightarrow$ $x[n]$ is WSS

$\not\Longleftarrow$



WSS Random Processes

DT RP $X[n]$ is WSS $n \in \mathbb{Z}$

iff

a) Mean Sequence
$$\mu_X[n] = E\big[X[n]\big] = \mu_X \rightarrow \text{constant}$$

b) Covariance Sequence
$$C_X[n_1, n_2] = E\big[X[n_1]\,X[n_2]\big] - \mu_X^2$$
$$= g(|n_2 - n_1|)$$
$$E\big[X[n_1]\,X[n_2]\big] = h(|n_2 - n_1|)$$

CT RP $X(t)$ is WSS $t \in \mathbb{R}$

iff

a) Mean Function

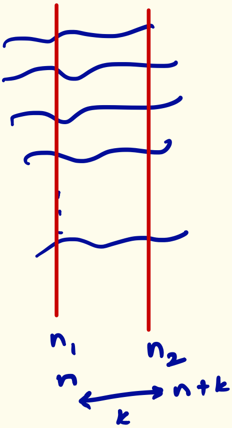
$$\mu_X(t) = E\big[X(t)\big] = \mu_X \rightarrow \text{constant}$$

b) Covariance Function

$$C_X(t_1, t_2) = E\big[X(t_1)\,X(t_2)\big] - \mu_X^2$$
$$= g(|t_2 - t_1|)$$
$$E\big[X(t_1)\,X(t_2)\big] = h(|t_2 - t_1|)$$

Auto Correlation Sequence (ACS)

If $x[n]$ is WSS, $E[x[n_1]\,x[n_2]] = h(|n_2 - n_1|)$



$n_1 = n$, $n_2 = n+k$

$$E[x[n_1]\,x[n_2]] = E[x[n]\,x[n+k]]$$

$$= r_x[n+k-n] = r_x[k]$$

$r_x[k] \longrightarrow$ ACS $\quad k \in \mathbb{Z}$

→ depends only on time difference between the samples $|n_2 - n_1| = |n+k-n|$
$= |k|$

→ Measures the correlation between 2 samples $(n, n+k)$
or r.vs at

→ value of n is arbitrary.

eg: Differencer

$$x[n] = u[n] - u[n-1] \qquad u[n] \to \text{IID}$$

Mean μ_u

Variance σ_u^2

$$E[u[k]\,u[l]] = \sigma_u^2\,\delta[l-k]$$

$$\mu_x[n] = E[x[n]] = E[u[n]] - E[u[n-1]]$$
$$= \mu_u - \mu_u = 0$$

$$r_x[k] = E\big[x[n]\,x[n+k]\big]$$
$$= E\big[(u[n] - u[n-1])(u[n+k] - u[n+k-1])\big]$$

$$E[u[k]\,u[l]] = \sigma_u^2\,\delta[l-k]$$

$$r_x[k] = E[u[n]\,u[n+k]] + E[u[n-1]\,u[n+k-1]]$$
$$- E[u[n-1]\,u[n+k]] - E[u[n]\,u[n+k-1]]$$

$$= \sigma_u^2\,\delta[k] + \sigma_u^2\,\delta[k] - \sigma_u^2\,\delta[k+1]$$
$$- \sigma_u^2\,\delta[k-1]$$

$$r_x[k] = 2\sigma_u^2\,\delta[k] - \sigma_u^2\,\delta[k-1] - \sigma_u^2\,\delta[k+1]$$

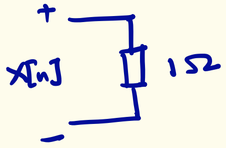
$$r_x[k] = \begin{cases} 2\sigma_u^2 , & k = 0 \\ -\sigma_u^2 , & |k| = 1 \\ 0 , & |k| > 1 \end{cases}$$

Properties of ACS $r_x[k]$

a) ACS for zero lag $^{k=0}$ $r_x[0] > 0$.

Proof

$$r_x[k] = E\left[x[n]\,x[n+k]\right]$$



$$r_x[0] = E\left[x^2[n]\right] > 0$$

x[n] → Voltage across a 1Ω resistor.

$r_x[0]$ → average power of x[n] ∀ n

$$y[n] = x^2[n]$$

Average power $E\left[x^2[n]\right] = r_x[0]$

b) ACS is an even sequence $r_x[k] = r_x[-k]$

Proof

$$r_x[k] = E\left[x[n]\,x[n+k]\right]$$

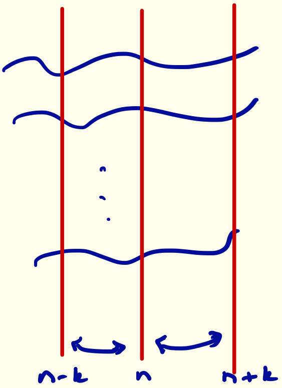
$n+k=m$ $\quad n=m-k$

$$= E\left[x[m-k]\,x[m]\right]$$

$$= E\left[x[m]\,x[m-k]\right]$$

$m+(-k)$

$$= r_x[-k]$$



c) Maximum absolute value of $r_x[k]$ is at $k=0$

$$|r_x[k]| \le r_x[0]$$

for some k

$$|r_x[k]| = r_x[0]$$

Proof Cauchy Schwarz Inequality

$$\left|E_{vw}[vw]\right| \le \sqrt{E_v[v^2]}\sqrt{E_w[w^2]}$$ $v, w \rightarrow$ r.vs.

$v = x[n]$, $w = x[n+k]$

$$\Big|E\big[x[n]\; x[n+k]\big]\Big| \le \sqrt{E[x^2[n]]}\;\sqrt{E[x^2[n+k]]}$$

$$\le \sqrt{r_x[0]}\;\sqrt{r_x[0]}$$

$$|r_x[k]| \le |r_x[0]|$$

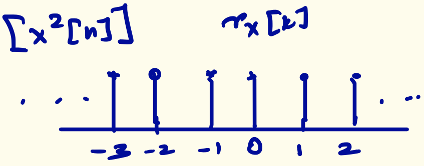
$$|r_x[k]| \le r_x[0]$$

If $x[n+k] = c\,x[n]$ $\forall n$ $c \rightarrow$ any scalar >0

$$\big|E[x[n]\;x[n+k]]\big| = \big|E[x[n]\;c\,x[n]]\big| = c\,E[x^2[n]]$$

$$\sqrt{E[x^2[n]]}\;\sqrt{E[c^2x^2[n]]} = c\,E[x^2[n]]$$

$$r_x[k] = r_x[0] \quad \forall k$$



d) ACS measures predictability of $X[n]$.

$X[n] \rightarrow$ zero mean R.P.

Correlation Coefficient $\rho_{X[n], X[n+k]} = \dfrac{r_x[k]}{r_x[0]}$

Proof $V, W \rightarrow$ zero mean r.v.s.

$$\rho_{V,W} = \frac{\operatorname{Cov}(V,W)}{\sqrt{\operatorname{var}(V)}\sqrt{\operatorname{var}(W)}} = \frac{E_{VW}[VW]}{\sqrt{E_V[V^2]}\sqrt{E_W[W^2]}}$$

$V = X[n]$, $W = X[n+k]$

$$\rho_{X[n], X[n+k]} = \frac{E[X[n]\,X[n+k]]}{\sqrt{E[X^2[n]]}\sqrt{E[X^2[n+k]]}}$$

$$= \frac{r_x[k]}{\sqrt{r_x[0]}\sqrt{r_x[0]}} \leftarrow \text{from (a)}$$

$$= \frac{r_x[k]}{r_x[0]}$$

eg: Differencer

$$\rho_{X[n], X[n+k]} = \begin{cases} 1, & k=0 \\ -\frac{1}{2}, & |k|=1 \\ 0, & |k|>1 \end{cases}$$

e) ACS approaches μ^2 as $k \to \infty$

$$r_x[k] = C_x[n, n+k] + \mu^2$$

As $k \to \infty$, $C_x[n, n+k] \to 0$

$$\Rightarrow r_x[k] \longrightarrow \mu^2$$

f) ACS is a positive semi-definite sequence.

Sample $\overset{RP}{x[n]}$ at $n = 0, 1, 2, \dots, k-1$

$$\vec{\underline{X}} = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[k-1] \end{bmatrix} \qquad x[n] \to \text{zero mean WSS}$$

$$\text{var}\left(\vec{a}^T \vec{\underline{X}}\right) = \vec{a}^T R_{\vec{\underline{X}}} \vec{a} \geq 0 \quad \text{for } k \geq 1$$

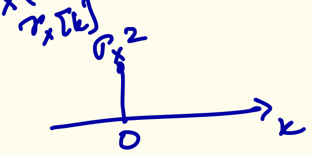
$$R_{\vec{\underline{X}}} = \begin{bmatrix} r_x[0] & r_x[1] & \dots & r_x[k-1] \\ r_x[1] & r_x[0] & \dots & r_x[k-2] \\ \vdots & & \ddots & \vdots \\ r_x[k-1] & r_x[k-2] & \dots & r_x[0] \end{bmatrix} = C_{\vec{\underline{X}}}$$

as $r_x[k] = C_x[n, n+k]$ $\qquad C_{\vec{\underline{X}}} = R_{\vec{\underline{X}}}$

eg. White Noise

$x[n] \rightarrow$ WSS R.P with zero mean, identical variance, uncorrelated samples

$$r_x[k] = E\left[x[n]\, x[n+k]\right] \qquad C_x(n,n+k) = r_x[k] - \mu_x^2$$



$$r_x[k] = \begin{cases} 0, & |k| > 0 \\ \sigma_x^2, & k = 0 \end{cases}$$

$$\delta[k] \doteq \begin{cases} 1, & k=0 \\ 0, & k \neq 0 \end{cases}$$

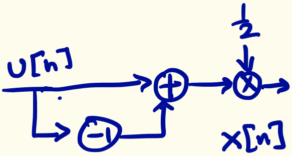
$$r_x[k] = \sigma_x^2\, \delta[k]$$

Finite Impulse Response

Causal

eg. Moving Average RP FIR $x[n] = \frac{1}{2}\left[u[n] + u[n-1]\right]$

$$C_x[n_1,n_2] = \begin{cases} \frac{\sigma_u^2}{2}, & n_1 = n_2 \\ \frac{\sigma_u^2}{4}, & |n_2 - n_1| = 1 \\ 0, & \text{o/w} \end{cases} \qquad n_2 - n_1 = k$$



$$r_x[k] = \begin{cases} \frac{\sigma_u^2}{2}, & k = 0 \\ \frac{\sigma_u^2}{4}, & |k| = 1 \\ 0, & |k| > 1 \end{cases}$$

$$r_x[k] = \frac{\sigma_u^2}{2}\delta[k] + \frac{\sigma_u^2}{4}\delta[k-1] + \frac{\sigma_u^2}{4}\delta[k+1]$$

eg: Auto-Regressive RP

$$X[n] = a\,X[n-1] + U[n] \qquad -\infty < n < \infty$$

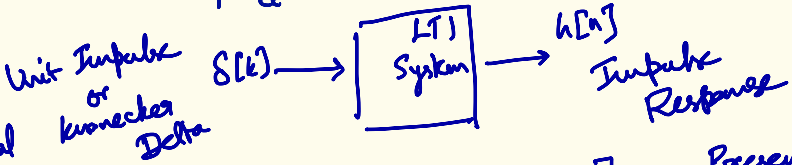
$|a| < 1$

$U[n] \rightarrow$ WGN zero mean

variance σ_U^2

IID

ACS $r_x[k] = \dfrac{\sigma_U^2}{1-a^2}\, a^{|k|} \quad \forall k$



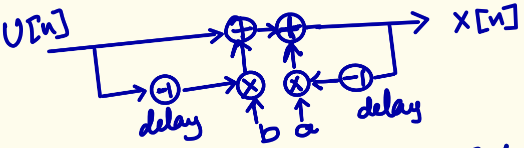
In general

FIR MA RP $X[n] = b\,U[n-1] + U[n]$ Present and Past inputs

IIR

AR RP $X[n] = a\,X[n-1] + U[n]$ Present input and Past output

ARMA RP



$$X[n] = a\,X[n-1] + b\,U[n-1] + U[n]$$

Present input
Past input
Past output

FIR → Finite Impulse Response

IIR → Infinite Impulse Response

Power Spectral Density (PSD)

$|X(f)|^2 = X^*(f)\,X(f)$

For any WSS DT RP $x[n]$, PSD $P_x(f)$

$$P_x(f) = \lim_{M\to\infty} \frac{1}{2M+1} E\left[\left| \overbrace{\sum_{n=-M}^{M} x[n]\, e^{-j2\pi f n}}^{X(f)} \right|^2 \right]$$

$$= \lim_{M\to\infty} \frac{1}{2M+1} E\left[\sum_{n=-M}^{M} x[n]\, e^{j2\pi f n} \sum_{m=-M}^{M} x[m]\, e^{-j2\pi f m} \right]$$

$$= \lim_{M\to\infty} \frac{1}{2M+1} \sum_{n=-M}^{M} \sum_{m=-M}^{M} E\left[x[n]\, x[m]\right] e^{-j2\pi f(m-n)}$$

$$= \lim_{M\to\infty} \frac{1}{2M+1} \sum_{n=-M}^{M} \sum_{m=-M}^{M} r_x[m-n]\, e^{-j2\pi f(m-n)}$$

We have

$$\sum_{n=-M}^{M} \sum_{m=-M}^{M} g(m-n) = \sum_{k=-2M}^{2M} (2M+1-|k|)\, g(k)$$

↘ Both sum of elements of $2M+1 \times 2M+1$ matrix

$$P_x(f) = \lim_{M\to\infty} \frac{1}{2M+1} \sum_{k=-2M}^{2M} (2M+1-|k|)\, r_x[k]\, e^{-2\pi f k}$$

$$P_x(f) = \lim_{M\to\infty} \sum_{k=-2M}^{2M} \left(1 - \frac{|k|}{2M+1}\right) r_x[k]\, e^{-j2\pi f k}$$

$$P_x(f) = \sum_{k=-\infty}^{+\infty} r_x[k]\, e^{-j2\pi f k}$$

DTFT of ACS $r_x[k]$

$$P_x(f) = \sum_{k=-\infty}^{+\infty} r_x[k]\, e^{-j2\pi f k}$$ Wiener-Khinchin theorem

Discrete Time Fourier Transform

$P_x(f)$ always exists as $\sum_{k=-\infty}^{+\infty} |r_x[k]| < \infty$

PSD is DTFT of ACS
$P_x(f)$ $r_x[k]$.

eg White Noise

$$r_x[k] = \sigma^2 \delta[k]$$

$$P_x(f) = \sum_{k=-\infty}^{+\infty} \sigma^2 \delta[k]\, e^{-j2\pi f k}$$

digital frequency

$$= \sigma^2 \qquad -\frac{1}{2} \le f \le \frac{1}{2}$$

$\omega = 2\pi f$

eg. AR RP

$$r_x[k] = \frac{\sigma_u^2}{1-a^2}\, a^{|k|} \qquad -\infty < k < \infty$$

$$P_x(f) = \frac{\sigma_u^2}{1+a^2-2a\cos(2\pi f)} \qquad -\frac{1}{2} \le f \le \frac{1}{2}$$

Properties of PSD $P_x(f)$

a) PSD is a real function.

$$P_x(f) = \sum_{k=-\infty}^{+\infty} r_x[k] \cos(2\pi f k)$$

Proof

$$P_x(f) = \sum_{k=-\infty}^{+\infty} r_x[k] \left(\cos(2\pi f k) - j \sin(2\pi f k)\right)$$

$$= \sum_{k=-\infty}^{+\infty} \underset{\text{even}}{r_x[k]} \underset{\text{even}}{\cos(2\pi f k)} - j \sum_{k=-\infty}^{+\infty} \underset{\text{even}}{r_x[k]} \underset{\text{odd}}{\sin(2\pi f k)}$$

even odd

$$\sum_{k=-\infty}^{+\infty} r_x[k] \sin(2\pi f k) = \sum_{k=-\infty}^{-1} r_x[k] \sin(2\pi f k) + \sum_{k=1}^{\infty} r_x[k] \sin(2\pi f k)$$

$$= \sum_{l=1}^{\infty} r_x[-l] \sin(-2\pi f l) + \sum_{k=1}^{\infty} r_x[k] \sin(2\pi f k)$$

$$= -\sum_{k=1}^{\infty} r_x[k] \sin(2\pi f k) + \sum_{k=1}^{\infty} r_x[k] \sin(2\pi f k)$$

$$= 0$$

$$\Rightarrow \quad P_x(f) = \sum_{k=-\infty}^{+\infty} r_x[k] \cos(2\pi f k)$$

DTCT Discrete Time Cosine Trans

b) PSD is non-negative

$$P_x(f) \geq 0$$

Proof

$$P_x(f) = \lim_{M\to\infty} \frac{1}{2M+1} E\left[\left|\sum_{n=-M}^{M} x[n]\, e^{-j2\pi fn}\right|^2\right] \geq 0$$

c) PSD is symmetric about f = 0. even function of f.

$$P_x(-f) = P_x(f)$$

Proof

$$P_x(f) = \sum_{k=-\infty}^{+\infty} r_x[k] \cos 2\pi fk$$

$$P_x(-f) = \sum_{k=-\infty}^{+\infty} r_x[k] \cos 2\pi fk$$

d) PSD is periodic with period one.

$$P_x(f+1) = P_x(f) \qquad f = 1$$

Proof

$$P_x(f+1) = \sum_{k=-\infty}^{+\infty} r_x[k] \cos(2\pi(f+1)k)$$

$$= \sum_{k=-\infty}^{+\infty} r_x[k] \cos(2\pi fk + 2\pi k) \qquad k \in \mathbb{Z}$$

$$= \sum_{k=-\infty}^{+\infty} r_x[k] \cos(2\pi fk)$$

$$= P_x(f)$$

e) ACS $r_x[k]$ from PSD $P_x(f)$.

IDTFT
$$r_x[k] = \int_{-\frac{1}{2}}^{\frac{1}{2}} P_x(f)\, e^{j2\pi fk}\, df \qquad -\infty < k < \infty$$

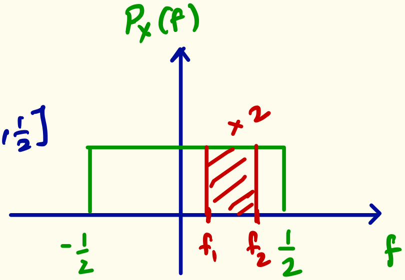
IDTCT
$$r_x[k] = \int_{-\frac{1}{2}}^{\frac{1}{2}} \underset{\text{even}}{P_x(f)}\, \underset{\text{even}}{\cos(2\pi fk)}\, df = 2\int_{0}^{\frac{1}{2}} P_x(f) \cos 2\pi fk\, df$$

f) Average power over a band of frequencies.

Average power in $0 \le f_1 \le f \le f_2$

Average physical power in $[f_1, f_2]$

$$= 2\int_{f_1}^{f_2} P_x(f)\, df$$



Average physical power in $[0, \frac{1}{2}]$

$$= 2\int_{0}^{\frac{1}{2}} P_x(f)\, df$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} P_x(f) \cos 2\pi(0) f\, df = r_x[0] = E[X^2[n]]$$

$X(t) \rightarrow$ CT $\overset{\text{WSS}}{\text{R.P.}}$ $\quad -\infty < t < \infty$

Bandlimited $[-W, \; W]$ $\quad$ Hz

Nyquist Sampling Theorem $\quad F_s \geq 2W$ $\quad$ Hz

DT RP

$X[n] \longrightarrow$ Bandlimited with max. frequency

$$= \frac{W}{F_s} = \frac{W}{2W} = \frac{1}{2}$$

$\longrightarrow$ Bandlimited $\left[-\frac{1}{2}, \frac{1}{2}\right]$

$\omega = 2\pi f$

rad/sec

Continuous Time WSS Random Processes

$X(t) \quad -\infty < t < \infty \longrightarrow$ WSS RP

$t \in \mathbb{R}$

Mean function $\mu_X(t) = E[X(t)] = \mu \qquad -\infty < t < \infty$

$C_X(t_1, t_2) = g(|t_2 - t_1|)$, $\quad r_X(t_1, t_2) = C_X(t_1, t_2) + \mu^2$

$= h(|t_2 - t_1|)$

Auto Correlation Function (ACF)

$$r_X(\tau) = E\left[X(t)\; X(t+\tau)\right] \qquad -\infty < \tau < \infty$$

$\tau \in \mathbb{R}$

$E[X(t_1)\, X(t_2)] \longrightarrow$ depends only on $|t_2 - t_1|$

$\in \mathbb{R}$

Properties of ACF $r_x(\tau)$

a) ACF is +ve for zero lag $r_x(0) > 0$

$$r_x(0) = E[X^2(t)] \rightarrow \text{Total average power.}$$

b) ACF is an even function (Symmetric about $\tau = 0$)

$$r_x(-\tau) = r_x(\tau)$$

c) Max. value of ACF is at $\tau = 0$

$$|r_x(\tau)| \le r_x(0)$$

d) ACF measures predictability of $X(t)$.

zero mean
$\mu = 0$

Correlation coefficient

$$\rho_{X(t), X(t+\tau)} = \frac{r_x(\tau)}{r_x(0)}$$

$$r_x(\tau) = C_x(\tau) + \mu^2$$

e) ACF $r_x(\tau) \longrightarrow \mu^2$ as $\tau \rightarrow \infty$ $C_x(\tau) \rightarrow$

f) ACF is a positive semidefinite function.

Power Spectral Density $P_x(F)$

$$P_x(F) = \lim_{T\to\infty} \frac{1}{T} E\left[\left| \int_{-T/2}^{T/2} x(t)\, e^{-j2\pi F t} dt \right|^2 \right]$$

Wiener-khinchin Theorem

F → Analog (Hz) frequency

$$P_x(F) = \int_{-\infty}^{+\infty} r_x(\tau)\, e^{-j2\pi F\tau}\, d\tau$$

$-\infty < F < \infty$

CTFT

$$= \int_{-\infty}^{+\infty} r_x(\tau) \cos(2\pi F\tau)\, d\tau$$

CTCT

Average physical power in $[F_1, F_2]$

$$= 2\int_{F_1}^{F_2} P_x(F)\, dF$$

Properties of PSD $P_x(F)$

a) PSD is a real function of F.

$$P_x(F) = \int_{-\infty}^{+\infty} r_x(\tau) \cos(2\pi F\tau)\, d\tau \longrightarrow \text{real}$$

b) PSD is non-negative.

$$P_x(F) \geq 0$$

c) PSD is symmetric about $F=0$

$$P_x(-F) = P_x(F)$$

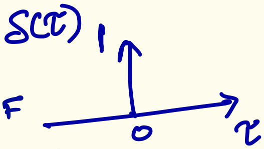
d) ACF $r_x(\tau)$ from $P_x(F)$

ICTFT

$$r_x(\tau) = \int_{-\infty}^{+\infty} P_x(F)\, e^{j2\pi F\tau}\, dF \qquad -\infty<\tau<\infty$$

ICTCT

$$r_x(\tau) = \int_{-\infty}^{+\infty} P_x(F) \cos(2\pi F\tau)\, dF$$



eg. White Gaussian Noise WGN $x(t)$

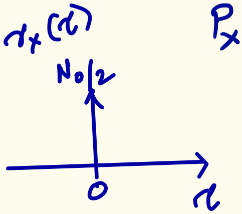
Dirac Delta (impulse) $\delta(\tau) = \begin{cases} \text{undefined}, & \tau=0 \\ 0, & \tau\neq 0 \end{cases}$

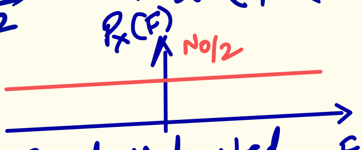
$$\int_{-\infty}^{+\infty} \delta(\tau)\, d\tau = 1$$

$$r_x(\tau) = \frac{N_0}{2}\, \delta(\tau)$$

$$\int_{-\infty}^{+\infty} \frac{N_0}{2}\, \delta(\tau)\, e^{-j2\pi F\tau}\, d\tau = \frac{N_0}{2}\int_{-\infty}^{+\infty} \delta(\tau)\, d\tau = \frac{N_0}{2}$$

$$P_x(F) = \frac{N_0}{2} \qquad -\infty < F < \infty$$





Band Unlimited

CT WSS RP $\quad r_x(\tau)$

$X(t) \longrightarrow$ Band limited if $P_x(F)$ exists for only $[-W, W]$, $W < \infty$

$r_x(\tau)$

Band unlimited if $P_x(F)$ exists for $(-\infty, \infty)$

DT WSS RP $\quad r_x[k]$

$X[n] \longrightarrow$ Band limited if $P_x(f)$ exists for only $[-W, W]$, $W < \frac{1}{2}$

$r_x[k]$

Band unlimited if $P_x(f)$ exists for $[-\frac{1}{2}, \frac{1}{2}] \rightarrow$ Full period

$X(t)$ or $x[n]$

$r_x(\tau)$ or $r_x[k]$ $\qquad$ $P_x(F)$ or $P_x(f)$

Time Limited $\Longrightarrow$ Band unlimited.

Time Unlimited $\Longleftarrow$ Band Limited

$-\infty < t < \infty$

eg: $P_x(f) = \sigma^2, \; -\frac{1}{2} \leq f \leq \frac{1}{2}$ $\qquad$ Band unlimited $\quad$ WGN

$r_x[k] = \sigma^2 \delta[k]$ $\qquad$ Time limited

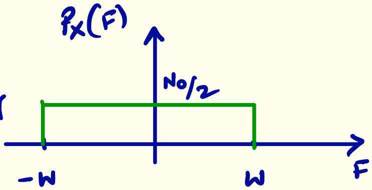
PSD

ACF $\quad r_x(\tau) \rightarrow$ Continuous Aperiodic $\Longrightarrow$ $P_x(F) \rightarrow$ Aperiodic Continuous

ACS $\quad r_x[k] \rightarrow$ Discrete Aperiodic $\Longrightarrow$ $P_x(f) \rightarrow$ Periodic Continuous eg. CT WSS RP $X(t)$

$$P_x(F) = \begin{cases} N_0/2 , & |F| \le W \\ 0 , & |F| > W \end{cases}$$

Band Limited



Spectrum

$$r_x(\tau) = \int_{-\infty}^{+\infty} P_x(F)\, e^{j2\pi F\tau}\, dF \quad \text{ICTFT}$$

$$= \frac{N_0}{2} \int_{-\infty}^{+\infty} e^{j2\pi F\tau}\, dF$$

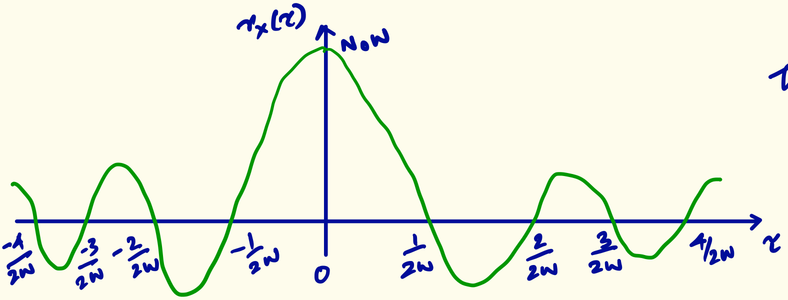
$$= \frac{N_0}{2} \int_{-W}^{W} \cos(2\pi F\tau)\, dF$$

$$= \frac{N_0}{2} \cdot 2 \int_{0}^{W} \cos(2\pi F\tau) dF$$

$$= N_0 \left. \frac{\sin(2\pi F\tau)}{2\pi\tau} \right|_0^W$$

$$= N_0 W \frac{\sin(2\pi W\tau)}{2\pi W\tau} \longrightarrow \text{Sinc function}$$

$$r_x(\tau) = N_0 W \ \text{sinc}\,(2\pi W\tau)$$

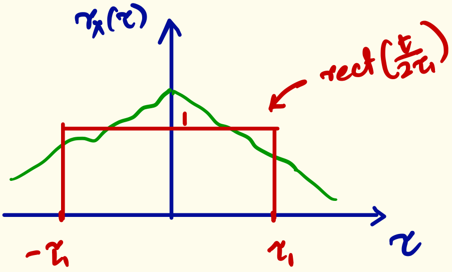


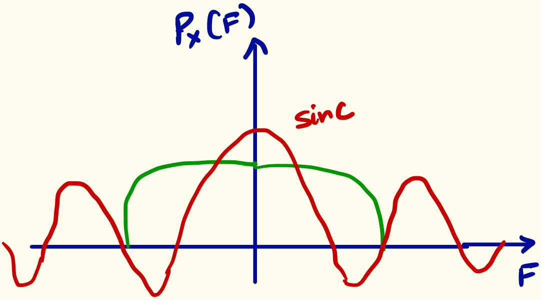
Time Unlimited

Multiplication in Time Domain ⟶ Convolution in Fourier Domain

$x_1(t)\, x_2(t)$ $\qquad$ $X_1(F) * X_2(F)$

Reverse of the property on 2 PMF/PDFs sum with their characteristic functions (Independent).





$$r_x'(\tau) = r_x(\tau) \times \text{rect}\left(\frac{\tau}{2\tau_1}\right) \qquad P_x'(F) = P_x(F) * \text{sinc}(F)$$

Time Limited $\quad$ Time Unlimited $\quad$ Time Limited $\quad$ Band Unlimited $\quad$ Band Limited $\quad$ Band Unlimited

Multiplication $\qquad$ Convolution

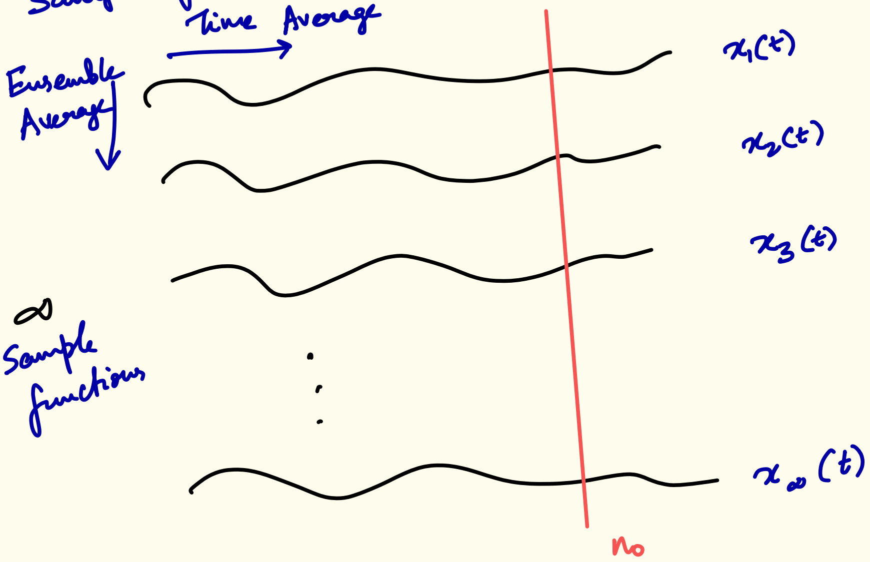
$$\begin{tikzcd} r_x(\tau) \arrow[r, shift left, "\text{CTFT}"] & P_x(F) \arrow[l, shift left, "\text{ICTFT}"] \end{tikzcd}$$

$$\begin{tikzcd} r_x'(\tau) \arrow[r, shift left, "\text{CTFT}"] & P_x'(F) \arrow[l, shift left, "\text{ICTFT}"] \end{tikzcd}$$

Ergodicity in Mean

Let $X[n]$ be a DT RP obtained by sampling a CT RP $X(t)$.

Sample functions of $X(t)$



Time Average $\mu_T = \frac{1}{N} \sum_{n=0}^{N-1} x_1[n]$ $\lim_{N \to \infty} var(\mu_T) = 0$

Ensemble Average $\mu_E = \lim_{M \to \infty} \frac{1}{M} \sum_{m=0}^{M-1} x_m[n_0]$ $n_0 \to$ fixed

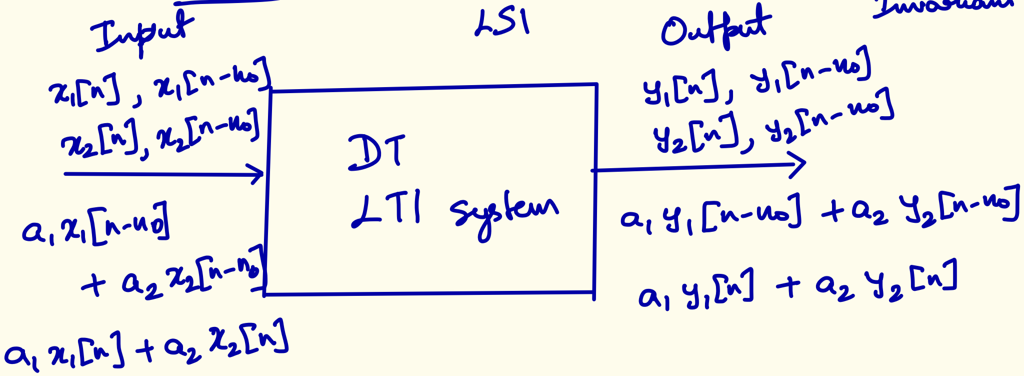
Ergodic in Mean $\mu_T = \mu_E$

eg. IID, WSS

LTI → Linear Time Invariant

Discrete Time LTI systems

LSI



Kronecker delta

Unit Impulse $\delta[n]$, $\delta[n-k]$

$$= \begin{cases} 1, & n=0 \\ 0, & o/w \end{cases}$$

$h[n]$ Impulse Response

$h[n-k]$

Any DT sequence/signal

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\ \delta[n-k]$$

ZS → zero state

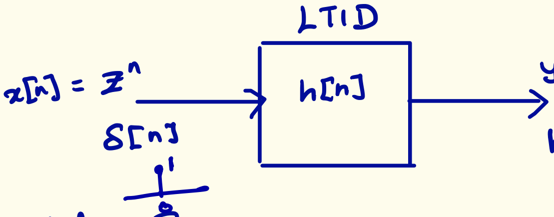
$$\Longrightarrow \quad y_{ZS}[n] = \sum_{k=-\infty}^{+\infty} x[k]\ h[n-k] \rightarrow \text{Discrete Convolution}$$

$$= \sum_{k=-\infty}^{+\infty} h[k]\ x[n-k]$$

DT LTI System

→ Characterized by impulse response $h[n]$

for all inputs $x[n]$



$y[n] = h[n] * z^n$
$z \in \mathbb{C}$

$h[n]$

Impulse Response

Zero-state Response

$$y[n] = \sum_{k=-\infty}^{+\infty} h[k]\, x[n-k] \qquad x[n] = z^n$$

$$= \sum_{k=-\infty}^{+\infty} h[k]\, z^{n-k}$$

$$= z^n \sum_{k=-\infty}^{+\infty} h[k]\, z^{-k}$$

$$y[n] = z^n \; H[z]$$

$$H[z] = \sum_{n=-\infty}^{+\infty} h[n]\, z^{-n} \longrightarrow \text{Transfer function}$$

If Causal $\quad h[n] = 0 \;,\; n < 0$

$$H[z] = \sum_{n=0}^{+\infty} h[n]\, z^{-n} \longrightarrow \text{Z-Transform of } h[n]$$

Constant Coeff. linear Difference Equation

$$y[n+N] + a_1\, y[n+N-1] + \dots + a_N\, y[n] = b_0\, x[n+N] + b_1\, x[n+N-1] + \dots + b_N\, x[n]$$

$$y[n+N] = E^N\, y[n]$$

$$Q[E]\, y[n] = P[E]\, x[n]$$

$$Q[E]\, z^n H[z] = P[E]\, z^n$$

$$a_1, a_2, \ldots, a_N = 0$$

$$\Rightarrow y[n+N] = b_0 x[n+N] + b_1 x[n+N-1] + \ldots + b_N x[n]$$

If $x[n] = \delta[n]$, $\quad y[n] = h[n]$

$$h[n+N] = b_0 \delta[n+N] + b_1 \delta[n+N-1] + \ldots + b_N \delta[n]$$

$h[n] \rightarrow$ finite length

$\rightarrow$ FIR filter.

$x[n] = z^n$

$y[n] = z^n H(z)$

Substituting in the difference equation

$$\left(z^{n+N} + a_1 z^{n+N-1} + \ldots + a_N z^n\right) H(z)$$

$$= b_0 z^{n+N} + b_1 z^{n+N-1} + \ldots + b_N z^n$$

Current sample $\rightarrow n+N$

$$\underbrace{(E^N + a_1 E^{N-1} + \cdots + a_N)}_{Q[E]} z^n H[z]$$

$$= \underbrace{(b_0 E^N + b_1 E^{N-1} + \cdots + b_N)}_{P[E]} z^n$$

$$\underbrace{(z^N + a_1 z^{N-1} + \cdots + a_N)}_{Q[z]} z^n H[z]$$

$$= \underbrace{(b_0 z^N + b_1 z^{N-1} + \cdots + b_N)}_{P[z]} z^n$$

$$Q[z] z^n H[z] = P[z] z^n$$

Transfer Function $H[z] = \left.\dfrac{P[z]}{Q[z]}\right|$ when $x[n] = z^n$

Z - Transform

Bilateral $X[z] = \displaystyle\sum_{n=-\infty}^{+\infty} x[n] z^{-n}$ non-causal

Inverse $x[n] = \dfrac{1}{2\pi j} \displaystyle\oint X[z] z^{n-1} dz$

Unilateral Z-Transform

$$X[z] = \sum_{n=0}^{\infty} x[n]\, z^{-n}$$ (Causal)

Existence

$$|X[z]| < \infty$$

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$$

$$= \frac{1}{1-x}, \quad |x| < 1$$

$$\Rightarrow \left| \sum_{n=0} x[n]\, z^{-n} \right| < \infty$$

$$\Rightarrow \sum_{n=0}^{\infty} \left| \frac{x[n]}{z^n} \right| < \infty$$

$$x[n] < \underline{r_0}^n \qquad \left| \frac{r_0}{z} \right| < 1$$

$$\sum_{n=0}^{\infty} \left| \frac{r_0}{z} \right|^n \qquad |z| > |r_0|$$

for any $x[n]$, if we can find r_0 such that

$$x[n] < r_0^n \Rightarrow \text{Z Transform exists}$$

Any -Infinity signal $x[n]$ growing not faster than exponential has an Z-Transform $X[z]$.

Any finite $x[n]$ $\longrightarrow$ Always has Z-Transform.

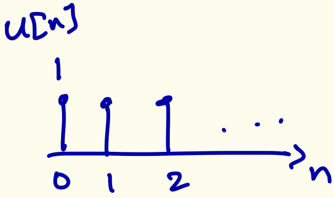
Eg.
Unit Impulse (shifted)

1. $\delta[n-k] \xrightarrow{Z} z^{-k}$

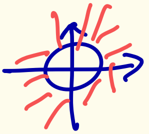
$$\delta[n-k] = \begin{cases} 1, & n=k \\ 0, & n \neq k \end{cases}$$

$$\sum_{n=0}^{\infty} \delta[n-k]\, z^{-n} = z^{-k}$$

2. Unit step $u[n] \xrightarrow{z} \frac{z}{z-1}$



$$\sum_{n=0}^{\infty} u[n]\, z^{-n} = \sum_{n=0}^{\infty} z^{-n} = \sum_{n=0}^{\infty} \left|\frac{1}{z}\right|^{n}$$

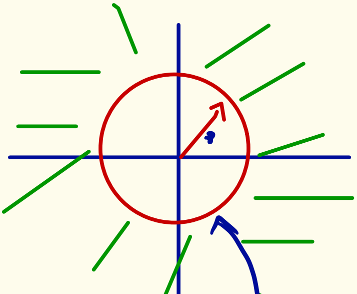


$\left|\frac{1}{z}\right| < 1$

$$= \frac{1}{1-\frac{1}{z}}$$

$|z| > 1$

$$= \frac{z}{z-1}$$



3. $h[n] = \gamma^{n} u[n] \xrightarrow{Z} \frac{z}{z-\gamma}$

$$\sum_{n=0}^{\infty} \gamma^{n} z^{-n} = \sum_{n=0}^{\infty} \left(\frac{\gamma}{z}\right)^{n}$$

ROC
$|z| > |\gamma|$

Converge when
$|\gamma| < 1$

$$= \frac{1}{1-\frac{\gamma}{z}}$$

$z=\gamma$
$\rightarrow$ Pole

will be within unit circle when $|\gamma| < 1$

$$= \frac{z}{z-\gamma} \longrightarrow H(z)$$

Properties

1. Addition

$$x_1[n] + x_2[n] \xrightarrow{Z} X_1[z] + X_2[z]$$

2. Scaling

$$a\,x[n] \xrightarrow{Z} a\,X[z]$$

$$\sum_{n=0}^{\infty} a\,x[n]\,z^{-n} = a \sum_{n=0}^{\infty} x[n] z^{-n} = a\,X[z]$$

3. Right Shifting

$$x[n-m]\,u[n-m] \xrightarrow{Z}$$

$$x\,[n-m]\,u[n] \xrightarrow{Z}$$

$$\sum_{n=m}^{\infty} x[n-m]\;z^{-n}$$

$n-m=r$

$$= \sum_{r=0}^{\infty} x[r]\;z^{-(r+m)}$$

$$= z^{-m} \sum_{r=0}^{\infty} x[r]\;z^{-r} \;=\; z^{-m}\,X[z]$$

eg.

$$x[n] \quad 3 \quad 4 \quad \underset{\overset{\uparrow}{0}}{5} \quad 6 \quad 7$$

$$x[n-2]\,u[n-2]$$

$$\underset{\overset{\uparrow}{0}}{0} \quad 0 \quad \underset{\uparrow}{5} \quad 6 \quad 7$$

$$x[n-2]\,u[n] \;\checkmark$$

$$\underset{\overset{\uparrow}{0}}{3} \quad 4 \quad 5 \quad 6 \quad 7$$

$$Z\Big[x[n-m]\,u[n]\Big] = \sum_{n=0}^{\infty} x\,[n-m]\,z^{-n}$$

$n-m=r$

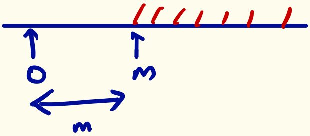
$n=m+r$

$$= \sum_{r=-m}^{\infty} x\,[r]\,z^{-(m+r)} = z^{-m}\left[\sum_{r=-m}^{-1} x[r]\,z^{-r} + \sum_{r=0}^{\infty} x[r]\,z^{-r}\right]$$

$$= z^{-m}\,X[z] + z^{-m} \sum_{r=-m}^{-1} x\,[r]\;z^{-r}$$

4. Left Shifting $x[n+m]\,u[n+m] \xrightarrow{Z} \underline{z^m X[z]}$

$x[n+m]\ u[n] \xrightarrow{Z}$



$\sum_{n=0}^{\infty} x[n+m]\, z^{-n}$ $\qquad n+m=r$

$= \sum_{r=m}^{\infty} x[r]\, z^{-(r-m)} = z^m\left[\sum_{r=0}^{\infty} x[r]\, z^{-r} - \sum_{r=0}^{m-1} x[r]\, z^{-r}\right]$

$= z^m X[z] - z^m \sum_{r=0}^{m-1} x[r]\, z^{-r}$

$x[n+2]\,u[n+2]$ $\qquad z^2 X[z]$

$0 \quad 0 \quad \underline{5 \quad 6} \quad \underset{\substack{\uparrow \\ 0}}{7}$

$x[n]\ \ \underset{-2}{3} \quad \underset{-1}{4} \quad \underset{\substack{\uparrow \\ 0}}{5} \quad \underset{1}{6} \quad \underset{2}{7}$

$x[n+2]\,u[n]$

$0 \quad 0 \quad 0 \quad 0 \quad \underset{\substack{\uparrow \\ 0}}{7}$

5. $\gamma^n x[n]\,u[n] \xrightarrow{Z} X\left[\frac{z}{\gamma}\right]$

$\sum_{n=0}^{\infty} \gamma^n x[n]\, z^{-n} = \sum_{n=0}^{\infty} x[n] \left(\frac{z}{\gamma}\right)^{-n}$ $\qquad x[n] \xrightarrow{Z} X[z]$

$= X\left[\frac{z}{\gamma}\right]$

6. $n\, x[n]\ u[n] \xrightarrow{Z} -z\frac{d}{dz}\left(X[z]\right)$

$-z\frac{d}{dz}\left(X[z]\right) = -z\frac{d}{dz}\left(\sum_{n=0}^{\infty} x[n]\, z^{-n}\right)$

$= -\underline{z} \sum_{n=0}^{\infty} -n\, x[n]\, z^{-n-1}$

$= \sum_{n=0}^{\infty} \underline{n\, x[n]}\ z^{-n} = Z\left[n x[n]\right]$

Time Convolution

7. $x_1[n] * x_2[n] \xrightarrow{Z} X_1[z]\, X_2[z]$

$$Z\left[x_1[n] * x_2[n]\right] = \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{+\infty} x_1[k]\, x_2[n-k] \right) z^{-n}$$

$n-k=m$

$n = m+k$

$$= \sum_{m=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} \underline{x_1[k]}\, x_2[m]\, \underline{z}^{-(m+\underline{k})}$$

$$= \sum_{k=-\infty}^{+\infty} x_1[k]\, z^{-k} \sum_{m=-\infty}^{+\infty} x_2[m]\, z^{-k}$$

$$= X_1[z]\, X_2[z]$$

Time Reversal

8. $x[-n] \xrightarrow{Z} X\left[\frac{1}{z}\right]$

$$\sum_{n=-\infty}^{+\infty} x[-n]\, z^{-n}$$

$n = -m$

$$= \sum_{m=-\infty}^{+\infty} x[m]\, z^{m} = \sum_{m=-\infty}^{+\infty} x[m] \left(\frac{1}{z}\right)^{-m}$$

$$= X\left[\frac{1}{z}\right]$$

Initial Value

$X[z] = x[0] + \frac{x[1]}{z} + \ldots$

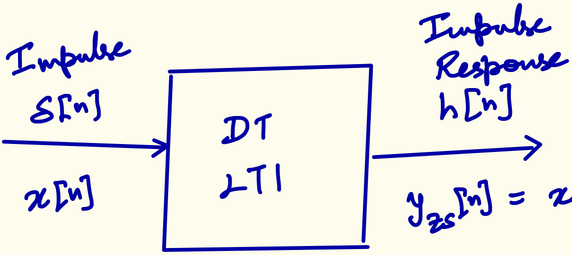
9. $x[0] = \lim_{z \to \infty} X[z]$

Final Value

10. $\lim_{N \to \infty} x[N] = \lim_{z \to 1} (z-1)\, X[z]$

<u>HW</u>

DT LTI System



$y_{zs}[n] = x[n] * h[n]$

Transfer function

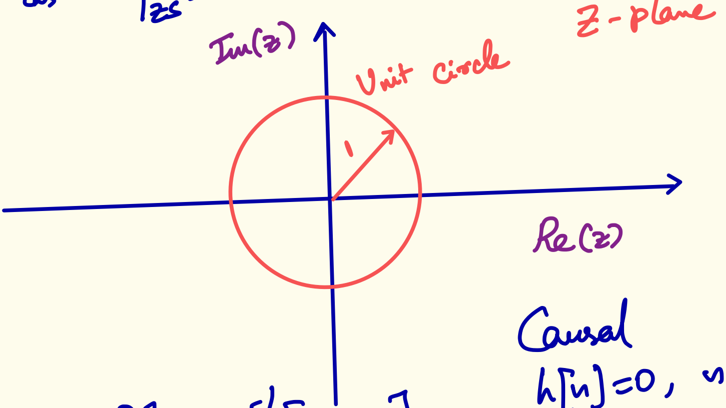
$$H[z] = \mathcal{Z}\left[h[n]\right]$$

$$H[z] = \frac{P[z]}{Q[z]} \rightarrow$$

(Const. coeff)
linear Difference
Equation
z-Transform of
P, Q → Coeff

$$H[z] = \frac{Y_{zs}[z]}{X[z]}$$

as $Y_{zs}[z] = H[z]\,X[z]$



Causal
$h[n] = 0, \quad n < 0$

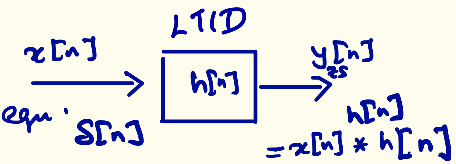
$$h[n] = \mathcal{Z}^{-1}\left[H[z]\right]$$

Stability

Transfer Function

$$H[z] = \frac{P[z]}{Q[z]}$$

Diff. equ.



$$H[z] = \sum_{n=0}^{\infty} h[n] z^{-n}$$ Impulse Response

$$Y_{zs}[z] = X[z]\, H[z] \xleftarrow{\mathcal{Z}}$$

$$H[z] = \frac{Y_{zs}[z]}{X[z]}$$ Zero state Response

Roots of $P[z]$ → zeros
Roots of $Q[z]$ → Poles
} of $H[z]$

No common roots
$P[z]$, $Q[z]$

LTID Causal System is

1. Asymptotically stable iff all the poles of $H[z]$ lie inside the unit circle.

2. Marginally stable iff all the poles of $H[z]$ lie inside the unit circle except unrepeated poles on the unit circle.

3. Unstable iff atleast one pole of $H[z]$ lies outside the unit circle or there are repeated poles on the unit circle.

Relation to DTFT

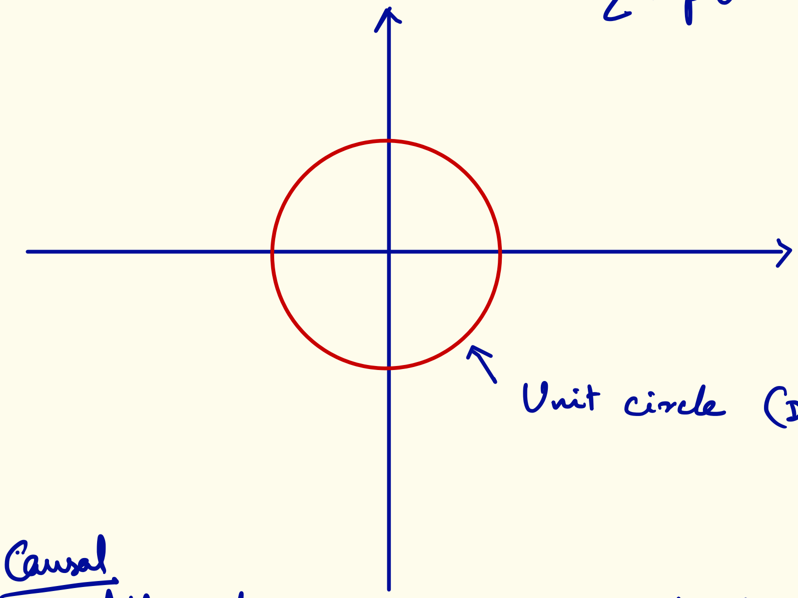
$$X(z) = \sum_{k=-\infty}^{+\infty} x[k]\, z^{-k} \qquad z = \underline{r}\, e^{j2\pi f} \qquad \text{Z-Transform}$$

$r=1$ Stable

$$X(f) = \sum_{k=-\infty}^{+\infty} x[k]\, e^{-j\,2\pi f k} \qquad \text{DTFT}$$

$$x[k] = \int_{-\frac{1}{2}}^{\frac{1}{2}} X(f)\, e^{j2\pi f k}\, df \qquad \text{IDTFT}$$

Z-plane



Unit circle (DTFT)

Causal

Stable

All poles inside Unit circle $\Rightarrow$ ROC includes unit circle
of $H[z]$

$\Rightarrow$ DTFT exists

eg.

1. $n\,u[n] \xrightarrow{\mathcal{Z}} \dfrac{z}{(z-1)^2}$

$$-z\,\frac{d}{dz}\left(\mathcal{Z}\left(u[n]\right)\right) = -z\,\frac{d}{dz}\left(\frac{z}{z-1}\right) = -z\,\frac{\left[(z-1)-z\right]}{(z-1)^2}$$

$$= \frac{z}{(z-1)^2}$$

2. $\gamma^n u[n] \xrightarrow{\mathcal{Z}}$

$$X\left[\frac{z}{\gamma}\right] = \frac{z/\gamma}{\frac{z}{\gamma}-1} = \frac{z}{z-\gamma}$$

3. $n\,\gamma^n u[n] \xrightarrow{\mathcal{Z}} -z\,\dfrac{d}{dz}\left(\dfrac{z}{z-\gamma}\right)$

$$= \frac{\gamma z}{(z-\gamma)^2}$$

4. $\cos\beta n\; u[n]$

$$= \left[\frac{e^{j\beta n}+e^{-j\beta n}}{2}\right]u[n]$$

$$= \frac{1}{2}\left[\left(e^{+j\beta}\right)^n u[n] + \left(e^{-j\beta}\right)^n u[n]\right]$$

$$\Big\downarrow \mathcal{Z}$$

$$\frac{1}{2}\left[\frac{z}{z-e^{j\beta}} + \frac{z}{z-e^{-j\beta}}\right]$$

HW

$n\cos\beta n\; u[n]$

$n\,\gamma^n\cos\beta n\; u[n]$

Eg. LTID system (Causal) $H[z] = \dfrac{3z+5}{z^2-5z+6} \xrightarrow{z^{-1}} h[n]$

$\dfrac{P[z]}{Q[z]} \leftarrow$

1\. $y[n+2] - 5y[n+1] + 6y[n] = 3x[n+1] + 5x[n]$

$\rightarrow y[-1] = 11/6 \;,\; y[-2] = \dfrac{37}{36} \;,\; x[n] = (0.5)^n u[n]$

$y[n] - 5y[n-1] + 6y[n-2] = 3x[n-1] + 5x[n-2]$

$$Y[z] - 5\left[\frac{1}{z}Y[z] + \frac{11}{6}\right] + 6\left[\frac{1}{z^2}Y[z] + \frac{11}{6z} + \frac{37}{36}\right] = z^{-1}\frac{3z}{z-0.5} + z^{-2}\frac{5z}{(z-0.5)} \quad ①$$

as $X[z] = \dfrac{z}{z-0.5}$

$y[n-2]\,u[n] \xrightarrow{Z} \dfrac{1}{z^2}Y[z] + \dfrac{1}{z}y[-1] + y[-2]$ $\quad m=2$

$y[n-1]\,u[n] \xrightarrow{Z} \dfrac{1}{z}Y[z] + y[-1]$ $\quad m=1$

$$x[n-m]\,u[n] \xrightarrow{Z} z^{-m}X[z] + z^{-m}\sum_{i=-m}^{-1} x[i]\,z^{-i}$$

$x[n] = (0.5)^n u[n] \xrightarrow{Z} \dfrac{z}{z-0.5}$

$x[n-2] = (0.5)^{n-2} u[n-2] \xrightarrow{Z} z^{-2}\dfrac{z}{z-0.5}$

$x[n-1] = (0.5)^{n-1} u[n-1] \xrightarrow{Z} z^{-1}\dfrac{z}{z-0.5}$

$$① \Rightarrow \left(Y[z] - \frac{5}{z}Y[z] + \frac{6}{z^2}Y[z]\right) + \left(\frac{11}{z} + \frac{37}{6} - \frac{55}{6}\right)$$

zero input

$$= \frac{3}{z-0.5} + \frac{5}{z(z-0.5)}$$

$$\left[\frac{z^2-5z+6}{z^2}\right]Y[z] = \left(-\frac{11}{z}+3\right) + \frac{3z+5}{z(z-0.5)}$$

$$Y_{zs}[z] = H[z]\,X[z]$$

$$Y[z] = \frac{z^2\left(-\frac{11}{z}+3\right)}{z^2-5z+6} + \frac{z^{\cancel{2}}(3z+5)}{\cancel{z}(z^2-5z+6)(z-0.5)} = \frac{3z+5}{z^2-5z+6}\cdot\frac{z}{z-0.5}$$

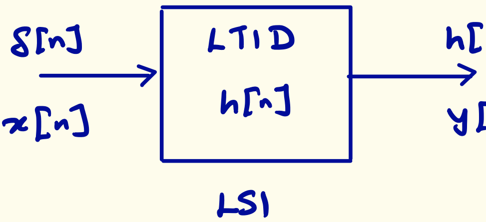
$$\frac{Y[z]}{z} = \frac{3z-11}{(z-3)(z-2)} + \frac{3z+5}{(z-3)(z-2)(z-0.5)}$$

$$\frac{Y[z]}{z} = \frac{5}{z-2} - \frac{2}{z-3} + \frac{5.6}{z-3} - \frac{22/3}{z-2} + \frac{1.733}{z-0.5}$$

$$Y[z] = \underbrace{\frac{5z}{z-2} - \frac{2z}{z-3}}_{Y_0[z]} + \underbrace{\frac{5.6z}{z-3} - \frac{22/3\,z}{z-2} + \frac{1.733z}{z-0.5}}_{Y_{zs}[z]}$$

$$y[n] = \underbrace{5(2)^n u[n] - 2(3)^n u[n]}_{y_0[n]} + \underbrace{5.6\ (3)^n u[n]}\ \underbrace{-\ 7.33\,(2)^n u[n] + 1.733\,(0.5)^n u[n]}_{y_{zs}[n]}$$

$$y[n] = \underbrace{-\,2.33\,(2)^n u[n] + 3.6\,(3)^n u[n]}_{y_n[n]\ \text{Natural Response}} + \underbrace{1.733\,(0.5)^n u[n]}_{y_\phi[n]\ \text{Forced Response}}$$



$h[n] \rightarrow$ Impulse Response

$y[n] = x[n] * h[n]$

$H(z) = Z[h[n]]$ → Transfer function

$$H(z) = \frac{P(z)}{Q(z)}$$

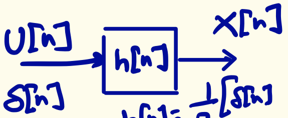
$Y(z) = Z[y[n]]$

$X(z) = Z[x[n]]$

$$H(z) = \frac{Y(z)}{X(z)}$$

Stable iff all roots of $Q(z)$ are within unit circle in z plane. $H(z) \equiv H(f)$

eg. Moving Average FIR



$h[n] = \frac{1}{2}[\delta[n] + \delta[n-1]]$

$$X[n] = \frac{1}{2}[U[n] + U[n-1]]$$

$U[n] \rightarrow$ WGN

Impulse Response $h[n] = \frac{1}{2}[\delta[n] + \delta[n-1]]$

$$Z[\delta[n]] = \sum_{n=-\infty}^{+\infty} \underset{\downarrow\, 1 \text{ at } n=0}{\delta[n]} z^{-n}$$

Transfer Function $H(z) = \frac{1}{2}[1 + z^{-1}] = \frac{1}{2}\left[\frac{z+1}{z}\right] = 1$

eg: Auto Regressive IIR

$$y[n] \xrightarrow{Z} Y(z)$$

$$y[n] = a\, y[n-1] + x[n]$$

o/p past o/p i/p

$$y[n-k] \xrightarrow{Z} z^{-k} Y(z)$$

$$Y(z) = a z^{-1} Y(z) + X(z)$$

$$(1 - a z^{-1}) Y(z) = X(z)$$

$$h[n] = Z^{-1}[H(z)]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - a z^{-1}}$$

eg. Difference FIR

$$y[n] = x[n] - x[n-1]$$

Impulse Response $h[n] = \delta[n] - \delta[n-1]$

$$Y(z) = X(z) - z^{-1} X(z)$$

Transfer Function $H(z) = \dfrac{Y(z)}{X(z)} = 1 - z^{-1}$

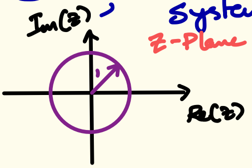
FIR → Pole at $z = 0$ ⟹ Stable

DT LTI Systems (Causal)

Impulse Response → h[n] → zero for $n<0$

Z-Transform

Transfer function $H(z) = \sum_{n=0}^{+\infty} h[n]\, z^{-n}$, $z = re^{j2\pi f}$, All systems



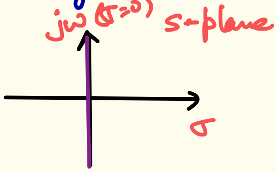
Inverse through look up table.

Discrete Time Fourier Transform (DTFT)

Transfer Function $H(f) = \sum_{n=0}^{+\infty} h[n]\, e^{-j2\pi fn}$, Unit circle $r=1$ in Z-Transform

$$h[n] = \int_{f=-\frac{1}{2}}^{+\frac{1}{2}} H(f)\, e^{j2\pi fn}\, df$$

Only stable Systems.



CT LTI systems (Causal)

Impulse Response → h(t) → zero for $t<0$

Laplace Transform

Transfer Function $H(s) = \int_{0}^{\infty} h(t)\, e^{-st}$, $s = \sigma + j\omega$, All systems

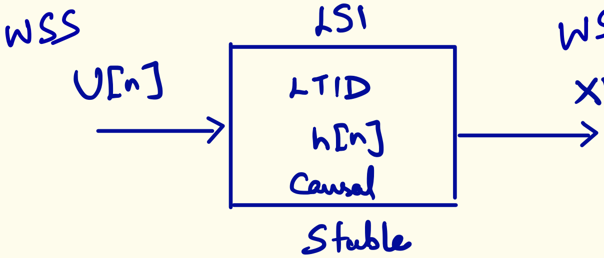
Inverse through table lookup.

Continuous Time Fourier Transform (CTFT)

Transfer Function $H(F) = \int_{0}^{\infty} h(t)\, e^{-j2\pi Ft}\, dt$, $\sigma = 0$ (jω axis), Only stable Systems

$$h(t) = \int_{-\infty}^{+\infty} H(F)\, e^{j2\pi Ft}\, dF$$

WSS Random Process through LTID System



$$X[n] = \sum_{k=-\infty}^{+\infty} h[k]\, U[n-k]$$

$U[n] \rightarrow$ WSS with mean μ_U and

ACS $r_U[k]$

$X[n] \rightarrow$ WSS with

$$\mu_x[n] = E\big[X[n]\big] = E\left[\sum_{k=-\infty}^{+\infty} h[k]\; U[n-k]\right]$$

$$= \sum_{k=-\infty}^{+\infty} h[k]\; E\big[U[n-k]\big] \cdot \underset{\downarrow}{1}$$

$$= \mu_U \sum_{k=-\infty}^{+\infty} h[k]\; \underbrace{\left(1 \cdot e^{-j2\pi(0)k}\right)}_{z^{-k}}$$

$$\mu_x = \mu_U\, H(1) \qquad z = 1 \Rightarrow 1\, e^{-j2\pi(0k)} = 1$$

H(0) in f domain

H(1) in z domain

$x[n] = h[n] * u[n]$

$$\gamma_x[k] = E\left[x[n]\, x[n+k]\right]$$

$$= E\left[\sum_{i=-\infty}^{+\infty} h[i]\, u[n-i] \sum_{j=-\infty}^{+\infty} h[j]\, u[n+k-j]\right]$$

$$= E\left[\sum_{i=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} h[i] h[j]\; u[n-i]\, u[n+k-j]\right]$$

$$= \sum_{i=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} h[i]\, h[j]\; E\left[u[n-i]\; u[n+k-j]\right]$$

$$= \sum_{i=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} h[i]\, h[j]\; \gamma_u\left[n+k-j-n+i\right]$$

$$= \sum_{i=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} h[i]\; h[j]\; \gamma_u\left[(k+i)-j\right]$$

$$= \sum_{i=-\infty}^{+\infty} h[i] \sum_{j=-\infty}^{+\infty} h[j]\; \gamma_u\left[(k+i)-j\right]$$

Let

$$\sum_{j=-\infty}^{+\infty} h[j]\; \gamma_u\left[(k+i)-j\right] = g[k+i]$$

$$r_x[k] = \sum_{i=-\infty}^{+\infty} h[i]\, g[k+i]$$

let $l = -i$

$$r_x[k] = \sum_{l=-\infty}^{+\infty} h[-l]\, g[k-l]$$

$$r_x[k] = h[-k] * g[k]$$

where $g[k] = h[k] * r_v[k]$

$$\Rightarrow \quad r_x[k] = h[-k] * h[k] * r_v[k]$$

Taking DTFT on both sides

$$\mathcal{F}[r_x[k]] = \mathcal{F}[h[-k]]\, \mathcal{F}[h[k]]\, \mathcal{F}[r_v[k]]$$

$$P_x(f) = H^*(f)\, H(f)\, P_v(f)$$

$$P_x(f) = |H(f)|^2 P_v(f)$$

eg: White Noise

$$P_v(f) = \sigma_v^2$$

$$P_x(f) = |H(f)|^2 \sigma_v^2$$

$$r_v[k] = \sigma_v^2 \delta[k]$$

$$r_x[k] = h[-k] * h[k] * \sigma_v^2 \delta[k]$$

$$= \sigma_v^2 h[-k] * h[k]$$

$$= \sigma_v^2 \sum_{i=-\infty}^{+\infty} h[-i]\, h[k-i]$$

Let $m = -i$

$$r_x[k] = \sigma_v^2 \sum_{m=-\infty}^{+\infty} h[m]\, h[k+m] \qquad -\infty < k < \infty$$

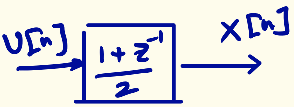
↖ Correlation

eg: Moving Average Random Process

$$x[n] = \left(v[n] + v[n-1]\right)/2$$

$v[n] \to$ White Noise

$$X(z) = \left(\frac{1+z^{-1}}{2}\right) V(z)$$



$$H(z) = \frac{X(z)}{V(z)} = \frac{1+z^{-1}}{2}$$

$$h[m] = \begin{cases} \frac{1}{2}, & m = 0, 1 \\ 0, & \text{o/w} \end{cases}$$

$$r_x[k] = \sigma_v^2 \sum_{m=0}^{1} h[m]\, h[m+k]$$

Convolution $y_1[n] = x_1[n] * x_2[n]$

Correlation $y_2[n] = x_1[n] \circledast x_2[n] = x_1[-n] * x_2[n]$

$$r_x[k] = \begin{cases} \sigma_v^2 \sum_{m=0}^{1} h^2[m] & , \quad k=0 \\ \sigma_v^2 \sum_{m=0}^{1} h[m]\, h[m+1] & , \quad |k|=1 \\ 0 & , \quad |k| \geq 2 \end{cases}$$

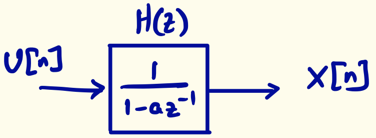
$$r_x[k] = \begin{cases} \sigma_v^2 \left[\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2\right] = \frac{\sigma_v^2}{2} & , \quad k=0 \\ \sigma_v^2 \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{\sigma_v^2}{4} & , \quad |k|=1 \\ 0 & , \quad |k| \geq 2 \end{cases}$$

$$P_x(f) = \sum_{k=-1}^{1} r_x[k]\, e^{-j2\pi f k}$$

$$= \frac{\sigma_v^2}{2} + \frac{\sigma_v^2}{4} e^{j2\pi f} + \frac{\sigma_v^2}{4} e^{-j2\pi f} = \frac{\sigma_v^2}{2} + \frac{\sigma_v^2}{2} \cos 2\pi f$$

$$P_x(f) = \frac{\sigma_v^2}{2}\left[1 + \cos 2\pi f\right] \qquad -\frac{1}{2} \leq f \leq \frac{1}{2}$$

eg: Auto Regressive RP



$$X[n] = a\, X[n-1] + U[n]$$

$U[n] \rightarrow$ WGN $\quad r_U[k] = \sigma_U^2 \delta[k]$

$P_U(f) = \sigma_U^2$

Taking z-transform

$$X(z) = a z^{-1} X(z) + U(z)$$

$$H(z) = \frac{X(z)}{U(z)} = \frac{1}{1-az^{-1}}$$

PSD $\quad P_X(f) = \left|H(e^{j2\pi f})\right|^2 \sigma_U^2 = \frac{\sigma_U^2}{\left|1-a e^{-j2\pi f}\right|^2}$

when $x[n] = \delta[n]$

$$h[n] = a\, h[n-1] + \delta[n]$$

$$H(z) = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$$

$$= \frac{1}{1-\frac{a}{z}} \quad |z|>|a|$$

$$h[-1] = 0$$

$$h[n] = a^n u_s[n] \qquad u_s[n] = \begin{cases} 0, & n<0 \\ 1, & n \geq 0 \end{cases}$$

for $k \geq 0$

$$r_X[k] = \sigma_U^2 \sum_{m=-\infty}^{+\infty} a^m u_s[m]\, a^{m+k} u_s[m+k] \longleftarrow \boxed{\begin{array}{l} h[-k] * \\ h[k] * r_U[k] \end{array}}$$

$$= \sigma_U^2 a^k \sum_{m=0}^{\infty} a^{2m} = \sigma_U^2 \frac{a^k}{1-a^2}, \quad |a|<1$$

$\forall k \quad r_X[k] = \sigma_U^2 \dfrac{a^{|k|}}{1-a^2}, \quad |a|<1$

IIR filter as $r_X[k] \neq 0$, as $k \rightarrow \infty$

Revision

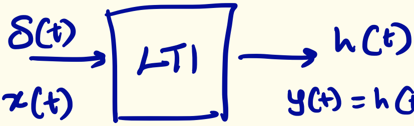
$x(t) \longrightarrow$ Continuous Time — Periodic ✗, Aperiodic ✓

$x[n] \longrightarrow$ Discrete Time — Periodic ✗, Aperiodic ✓

Fourier Series $\longrightarrow$ Periodic

Aperiodic

CT



$y(t) = h(t) * x(t)$

Const. Coeff. $Y(F) = H(F)\,X(F)$

Linear ODE $Y(s) = H(s)\,X(s)$

$$x(t) \xrightarrow{\text{CTFT}} X(F)$$

$$= \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi Ft}\, dt$$

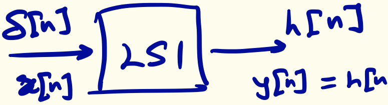
Imag. axis of s

$$h(t) \xrightarrow{\text{Laplace}} H(s)$$

$$= \int_{-\infty}^{+\infty} h(t)\, e^{-st}\, dt$$

$$s = \sigma + j2\pi F$$

DT



$y[n] = h[n] * x[n]$

$Y(f) = H(f)\,X(f)$

$Y(z) = H(z)\,X(z)$

Const. Coeff.

Difference Equation

$$x[n] \xrightarrow{\text{DTFT}} X(f)$$

$$= \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j2\pi fn}$$

unit circle of z

$$h[n] \xrightarrow{Z} H(z)$$

$$= \sum_{n=-\infty}^{+\infty} h[n]\, z^{n}$$

$$z = re^{j2\pi f}$$

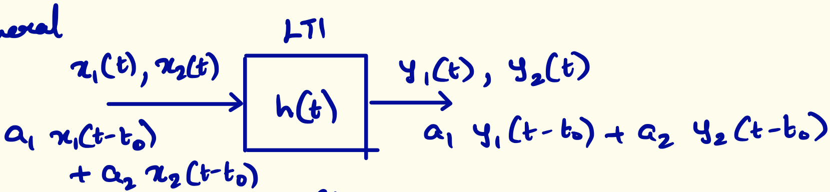
DTFT $P_x(f) = \sum_{k=-\infty}^{+\infty} r_x[k]\, e^{-j2\pi f k}$ $\qquad P_x(f+1) = P_x(f)$

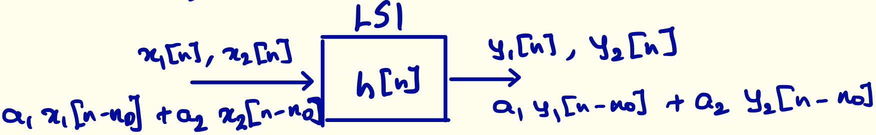
$$r_x[0] = E\left[x^2[n]\right]$$

IDTFT $r_x[k] = \int_{-\frac{1}{2}}^{\frac{1}{2}} P_x(f)\, e^{j2\pi f k}\, df \quad \leftarrow k=0$

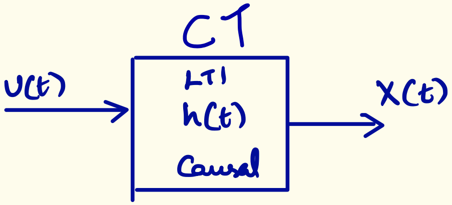
Time		Freq.
1. Periodic		1. Discrete
2. Aperiodic		2. Continuous
3. Discrete $r_x[k]$		3. Periodic $P_x(f)$
4. Continuous $r_x(\tau)$		4. Aperiodic $P_x(F)$
5. Time limited	$\Rightarrow$	5. Band unlimited
6. Time unlimited	$\Leftarrow$	6. Band limited

In general





For Continuous time WSS Random Process $U(t)$



$U(t)$ → mean μ_U and ACF $r_U(\tau)$

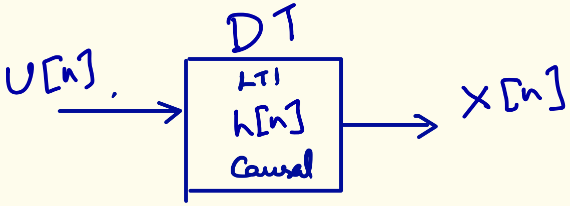
$X(t)$ → WSS with

Mean $\mu_X = H(0)\,\mu_U$

ACF $r_X(\tau) = h(-\tau) * h(\tau) * r_U(\tau)$

PSD $P_X(F) = |H(F)|^2 P_U(F)$

For discrete time WSS Random Process $U[n]$



$U[n] \rightarrow$ mean μ_U and ACS $r_U[k]$

$X[n] \rightarrow$ WSS with

Mean $\mu_X = H(1)\,\mu_U$, $H(z=1)$ or $H(f=0)$

ACS $r_X[k] = h[-k] * h[k] * r_U[k]$

PSD $P_X(f) = |H(f)|^2 P_U(f)$ stable

$P_X(z) = |H(z)|^2 P_U(z)$ any system

Gaussian Random Processes

DTCV

- Physically motivated by CLT
- Mathematically Tractable
- Joint PDF of any number of random variables is multi-variable Gaussian
- Given mean sequence & covariance sequence, we can estimate joint PDF
- If Gaussian RP is WSS, it is also SSS.
- Gaussian RP through an LTI system gives another Gaussian RP.

Multi-variate Gaussian PDF

$$\underline{\vec{X}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$$

Joint PDF

$$p_{\underline{\vec{X}}}(\vec{x}) = \frac{1}{(2\pi)^{N/2} \det^{\frac{1}{2}}(C_{\underline{\vec{X}}})} e^{-\frac{1}{2}(\vec{x}-\vec{\mu}_x)^T C_{\underline{\vec{X}}}^{-1} (\vec{x}-\vec{\mu}_x)}$$

$$\vec{\mu}_x = E_{\underline{\vec{X}}}[\underline{\vec{X}}] = \begin{bmatrix} E_{x_1}[x_1] \\ \vdots \\ E_{x_N}[x_N] \end{bmatrix}$$

Covariance Matrix N×N

$$C_{\underline{\vec{X}}} = \begin{bmatrix} var(x_1) & Cov(x_1, x_2) & \cdots & Cov(x_1, x_N) \\ Cov(x_1, x_2) & var(x_2) & \cdots & Cov(x_2, x_N) \\ \vdots & & \ddots & \vdots \\ Cov(x_1, x_N) & & \cdots & var(x_N) \end{bmatrix}$$

1. $\vec{\mu}_x$, $C_{\underline{\vec{X}}}$ define entire PDF.
2. If $C_{\underline{\vec{X}}}$ is diagonal $[C_{\underline{\vec{X}}}]_{i,j} = 0,\ i \neq j$ then $x_1, x_2, \ldots, x_N$ are uncorrelated and independent.

$$\vec{X} \sim N(\vec{\mu}_{\vec{X}}, C_{\vec{X}})$$

3. $$\vec{Y} = G\vec{X} \qquad G \rightarrow M \times N, \quad M \leq N$$

$$\vec{Y} \sim N(G\vec{\mu}_{\vec{X}}, G C_{\vec{X}} G^T)$$

eg: White Noise

$$x[n]$$

$$\mu_x[n] = E[x[N]] = 0$$

$$r_x[k] = \sigma^2 \delta[k]$$

White Gaussian Noise

$$x[n_i] \sim N(0, \sigma^2) \quad \text{each marginal PDF}$$

$$\vec{X} = \begin{bmatrix} x[n_1] \\ x[n_2] \\ \vdots \\ x[n_k] \end{bmatrix} \longrightarrow \text{IID}$$

$$p_{\vec{X}}(\vec{x}) = \prod_{i=1}^{K} p_{x[n_i]}(x[n_i])$$

$$= \frac{1}{(2\pi)^{k/2} \det^{\frac{1}{2}}(\sigma^2 I)} e^{-\frac{1}{2}\vec{x}^T(\sigma^2 I)^{-1}\vec{x}}$$

$$\vec{X} \sim N(\vec{0}, \sigma^2 I)$$

WGN is IID $\Rightarrow$ stationary.

a) Gaussian RP with uncorrelated random variables have independent random variables.

Proof $C_{\vec{X}} \rightarrow$ diagonal

$$p_{\vec{X}}(\vec{x}) = \prod_{i=1}^{k} p_{x[n_i]}(x[n_i]) \rightarrow \text{independent.}$$

b) A WSS Gaussian RP is also SSS.

$E[x[n_i]] = \mu$

$\text{Cov}(x[n_i], x[n_j]) = r_x[n_j - n_i] - \mu^2$

$i = 1, 2, \ldots, k$

Proof

$x[n] \rightarrow$ Gaussian WSS RP

$$E[x[n_i + n_0]] = \mu_x[n_i + n_0] = \mu$$

$$[C_{\vec{X}}]_{n_i+n_0, n_j+n_0} = \text{Cov}(x[n_i + n_0], x[n_j + n_0])$$

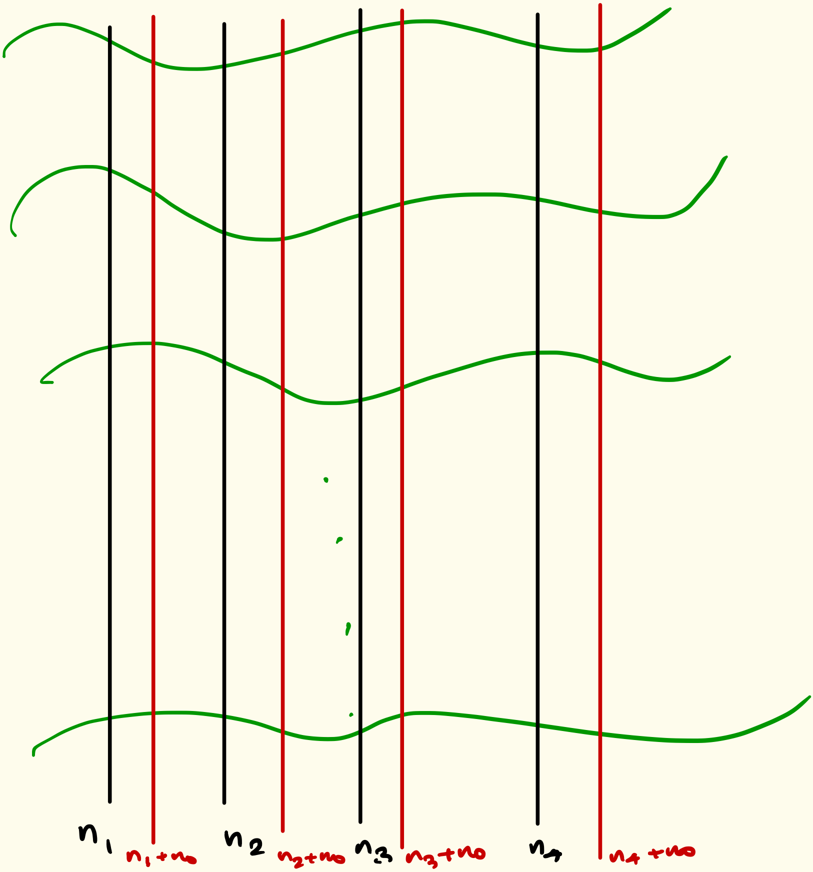
$$= E[x[n_i + n_0]\, x[n_j + n_0]] - \mu^2$$

$$= r_x[n_j - n_i] - \mu^2 \quad \text{for } i, j = 1, 2, \ldots, k$$

$$p_{\vec{X}[n_i]}(x[n_i]) \sim N(\vec{\mu}, C_{\vec{X}}), \quad i=1,2,\dots,k$$

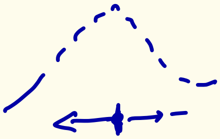
$$p_{\vec{X}[n_i+n_0]}(x[n_i+n_0]) \sim N(\vec{\mu}, C_{\vec{X}}), \quad i=1,2,\dots,k$$

$\Longrightarrow$ x[n] is SSS as Joint PDF is same with and without time shift.



eg. Discrete-Time Wiener RP or Brownian Motion

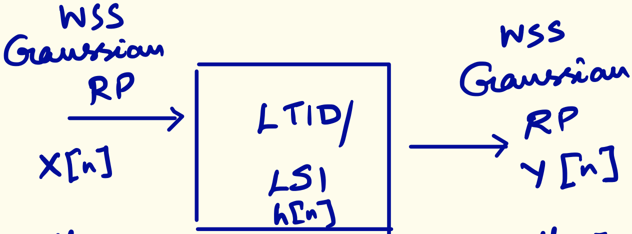
$U[n] \rightarrow$ WGN with σ_U^2



$$X[n] = \sum_{i=0}^{n} U[i] \quad , \quad n \geq 0$$

$\mu_X[n] = 0$

$\mathrm{var}\left(X[n]\right) = (n+1)\,\sigma_U^2$



μ_X

$r_X[k]$

$P_X(f)$

$\mu_Y = H(0)\,\mu_X$ $\quad H(z=0)$

$r_Y[k] = h[k] * h[-k] * r_X[k]$

$P_Y(f) = |H(f)|^2\, P_X(f)$

$X[n] \sim \mathcal{N}\left(\vec{\mu}_X, C_{\underline{X}}\right)$

$\uparrow$

$r_X[m-n] - \mu_X^2$

$Y[n] \sim \mathcal{N}\left(H(0)\vec{\mu}_X, \; r_Y[m-n] - \left(\mu_X H(0)\right)^2\right)$

$[C_Y]_{m,n} =$

$\sim \mathcal{N}\left(\vec{\mu}_Y, C_{\underline{Y}}\right)$

eg: Differencer

$X[n] \longrightarrow$ WSS Gaussian mean μ_x, ACS $r_x[k]$

$$Y[n] = X[n] - X[n-1] \overset{Z}{\Longrightarrow} Y(z) = X(z)(1 - z^{-1})$$

$$H(z) = 1 - z^{-1}$$

$$E[Y[n]] = E[X[n]] - E[X[n-1]] = \mu_x - \mu_x = 0$$

$$H(f) = H(z = e^{j2\pi f}) = 1 - e^{-j2\pi f}$$

$$P_y(f) = H^*(f)\, H(f)\, P_x(f)$$

$$= [1 - e^{j2\pi f}][1 - e^{-j2\pi f}]\, P_x(f) = [1 + 1 - e^{j2\pi f} - e^{-j2\pi f}]\, P_x(f)$$

$$= 2P_x(f) - e^{j2\pi f} P_x(f) - e^{-j2\pi f} P_x(f)$$

$$r_y[k] = 2r_x[k] - r_x[k+1] - r_x[k-1] \qquad \Big\uparrow \text{DTFT}$$

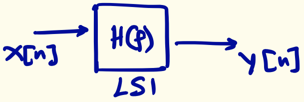
Let $\vec{y} = \begin{bmatrix} Y[0] \\ Y[1] \end{bmatrix}$

$$C_{\vec{y}} = \begin{bmatrix} r_y[0] & r_y[1] \\ r_y[1] & r_y[0] \end{bmatrix} = \begin{bmatrix} 2(r_x[0] - r_x[1]) & 2r_x[1] - r_x[2] - r_x[0] \\ 2r_x[1] - r_x[2] - r_x[0] & 2(r_x[0] - r_x[1]) \end{bmatrix}$$

Joint PDF

$$p_{Y[0],Y[1]}(y[0], y[1]) = \frac{1}{2\pi \det^{\frac{1}{2}}(C_{\vec{y}})}\, e^{-\frac{1}{2}\vec{y}^T C_{\vec{y}} \vec{y}}$$

In general,



$x[n] \rightarrow$ WSS Gaussian RP mean μ_x, ACS $r_x[k]$
(SSS)

PDF of N successive output samples

$$\vec{y} = \begin{bmatrix} y[0] \\ y[1] \\ \vdots \\ y[N-1] \end{bmatrix}$$

$$p_{\vec{y}}(\vec{y}) = \frac{1}{(2\pi)^{N/2} \det^{\frac{1}{2}}(C_{\vec{y}})} e^{-\frac{1}{2}(\vec{y} - \mu_{\vec{y}})^T C_{\vec{y}}^{-1} (\vec{y} - \mu_{\vec{y}})}$$

$$\mu_{\vec{y}} = H(0) \begin{bmatrix} \mu_x \\ \mu_x \\ \vdots \\ \mu_x \end{bmatrix}$$

$$\left[C_{\vec{y}}\right]_{mn} = r_y[m-n] - (\mu_x H(0))^2$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} |H(f)|^2 P_x(f) e^{j2\pi f(m-n)} df - (\mu_x H(0))^2$$

$m = 1, 2, \ldots, N$; $n = 1, 2, \ldots, N$

$y[n] \rightarrow$ WSS, SSS Gaussian RP

eg. $U[n] \rightarrow$ White Gaussian Noise (WGN)

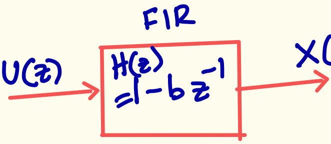
$\rightarrow$ WSS, SSS

a) Moving Average RP (MA)

$$X[n] = U[n] - b\,U[n-1]$$

Taking Z Transform $X(z) = U(z) - b z^{-1} U(z) \Rightarrow H(z) = \frac{X(z)}{U(z)}$

$= 1 - b z^{-1}$



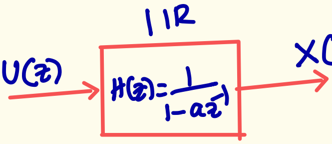
$X(z) = H(z)\,U(z)$

b) Auto Regressive RP (AR)

$$X[n] = a\,X[n-1] + U[n]$$

Taking Z transform $X(z) = a z^{-1} X(z) + U(z) \Rightarrow H(z) = \frac{X(z)}{U(z)}$

$= \frac{1}{1 - a z^{-1}}$



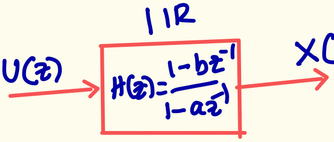
$X(z) = H(z)\,U(z)$

c) Auto Regressive Moving Average RP (ARMA)

$$X[n] = a\,X[n-1] - b\,U[n-1] + U[n]$$

Taking Z Transform $X(z) = a z^{-1} X(z) - b z^{-1} U(z) + U(z)$

$\Rightarrow H(z) = \frac{1 - b z^{-1}}{1 - a z^{-1}}$



$X(z) = H(z)\,U(z)$

All are Gaussian WSS/SSS Random Processes.

CTCV Gaussian RP

$X(t) \rightarrow$ CT Gaussian RP if

$$\vec{X} = \begin{bmatrix} X(t_1) \\ X(t_2) \\ \vdots \\ X(t_k) \end{bmatrix} \rightarrow \text{Multivariate Gaussian PDF}$$

$\forall \{t_1, t_2, \ldots, t_k\}$

$\forall k$

eg: CT WGN

$X(t) \rightarrow$ zero mean, ACF $r_X(\tau) = \frac{N_0}{2}\delta(\tau)$

$r_X(\tau) = 0 \quad \forall \tau \neq 0 \Rightarrow$ Uncorrelated

$\Rightarrow$ Independent (Gaussian)

eg: CT Wiener RP or Brownian Motion

$U(t) \rightarrow$ WGN $\sim N\left(0, \frac{N_0}{2}\right)$

$E[U(\tau)] = 0$

$\forall \tau$

$$X(t) = \int_0^t U(\tau)\, d\tau, \quad t \geq 0$$

$$E[X(t)] = E\left[\int_0^t U(\tau)\, d\tau\right]$$

$$= \int_0^t E[U(\tau)]\, d\tau = 0$$

$$E[X(t_1)X(t_2)] = E\left[\int_0^{t_1} v(\tau_1)\,d\tau_1 \int_0^{t_2} v(\tau_2)\,d\tau_2\right]$$

$$= \int_0^{t_1}\int_0^{t_2} E[v(\tau_1)\,v(\tau_2)]\,d\tau_1\,d\tau_2$$

$$= \int_0^{t_1}\int_0^{t_2} r_v(\tau_2-\tau_1)\,d\tau_1\,d\tau_2$$

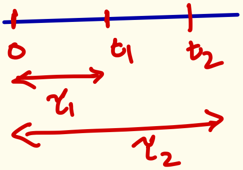
Here $r_v(\tau_2-\tau_1) = \frac{N_0}{2}\,\delta(\tau_2-\tau_1)$

$$= \frac{N_0}{2}\underset{\tau_1=0}{\int^{t_1}}\ \underset{\tau_2=}{\int_0^{t_2}} \delta(\tau_2-\tau_1)\,d\tau_2\,d\tau_1$$

Let $0 < t_1 < t_2$ $\quad \int_0^{t_2} \delta(\tau_2-\tau_1)\,d\tau_2$ $\quad$ fix τ_1

$0 \le \tau_2 \le t_2$

$= 1$ for $0 \le \tau_1 \le t_1$



$$E[X(t_1)X(t_2)] = \frac{N_0}{2}\int_0^{t_1} d\tau_1 = \frac{N_0}{2}\,t_1$$

If $0 < t_2 < t_1$
$$E[X(t_1)X(t_2)] = \frac{N_0}{2}\,t_2$$

$$E[X(t)] = 0\,, \quad E[X(t_1)X(t_2)] = \frac{N_0}{2}\min(t_1, t_2)$$

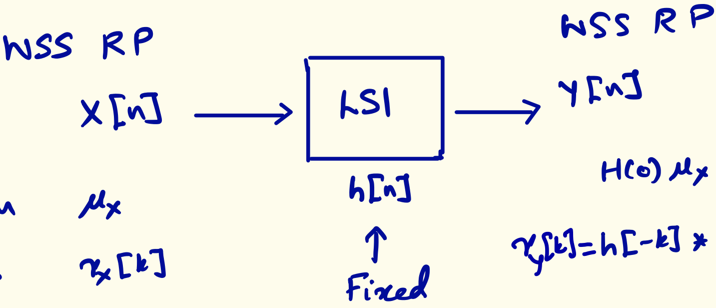
$t_1 = t_2 = t$

PDF of $X(t) \sim N\left(0, \frac{N_0}{2}t\right)$

→ Non-stationary

→ Marginal is Gaussian (Var increases with t)

Deterministic LTI | LSI system



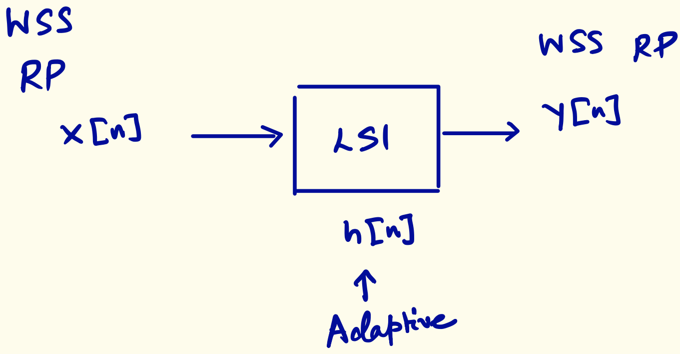
$H(0)\,\mu_x$

Mean	μ_x
ACS	$r_x[k]$
PSD	$P_x(f)$

$$r_y[k] = h[-k] * h[k] * r_x[k]$$

$$P_y(f) = |H(f)|^2 P_x(f)$$

Adaptive LTI / LSI system



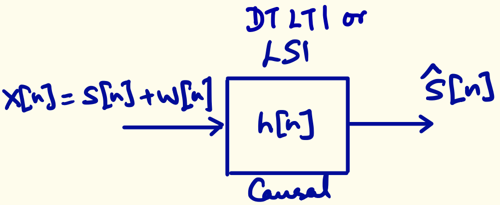
Impulse Response $h[n] \approx f\left(r_x[k]\right)$

Transfer Function $H(f) \approx f\left(P_x(f)\right)$

Wiener Filtering

a) Filtering

WSS



WSS

$h[n] = 0$ for $n < 0$

Input $\{ x[n_0],\ x[n_0-1],\ x[n_0-2],\ . \ . \ . \ \}$

RP $x[n]$ $\quad \uparrow$

Present $\qquad\qquad\qquad$ Past

$s[n] \rightarrow$ Desired Signal

$w[n] \rightarrow$ Noise signal

$$\hat{s}[n_0] = \sum_{k=0}^{\infty} h[k]\, x[n_0-k]$$

$h[n] = 0,\ n < 0$

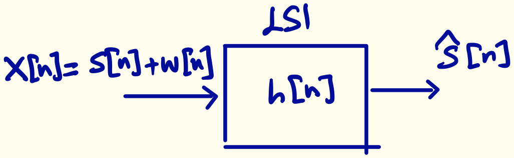
Causal

$\hat{s}[n_0]$ for all n_0

is a function of present and past inputs.

Real-time Implementation.

b) Smoothing



$-\infty < n < \infty$

Input $\{\ldots, x[n_0+2], x[n_0+1], x[n_0], x[n_0-1], \ldots\}$

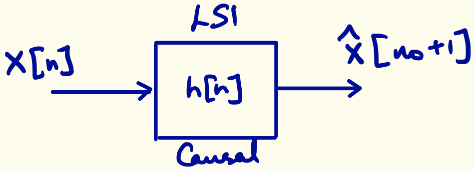
RP $x[n]$

$$\underbrace{\ldots, x[n_0+2], x[n_0+1]}_{\text{Future}},\ \underset{\substack{\uparrow \\ \text{Present}}}{x[n_0]},\ \underbrace{x[n_0-1], \ldots}_{\text{Past}}$$

$$\hat{s}[n_0] = \sum_{k=-\infty}^{+\infty} h[k]\, x[n_0-k]$$ Non-Causal

$\hat{s}[n_0]$ is a function of present, past and future inputs.

c) Prediction | Forecasting | Extrapolation



Input $\{x[n_0], x[n_0-1], x[n_0-2], \ldots\}$

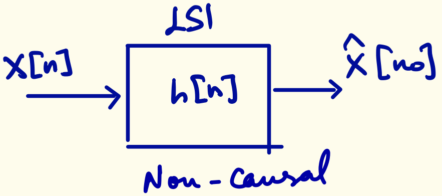
RP $x[n]$

$$\underset{\substack{\uparrow \\ \text{Present}}}{x[n_0]},\ \underbrace{x[n_0-1], x[n_0-2], \ldots}_{\text{Past}}$$

$$\hat{x}[n_0+1] = \sum_{k=0}^{\infty} h[k]\, x[n_0-k] \qquad \text{Causal}$$

$\hat{x}[n_0+1]$ is a function of present and past inputs.

d) Interpolation



Not known

Input RP $X[n]$ $\{\ldots, x[n_0+2], x[n_0+1], x[n_0], x[n_0-1], \ldots\}$

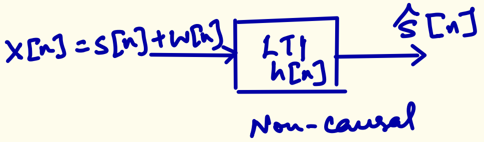
$\uparrow$

Future Present Past

$$\hat{x}[n_0] = \sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} h[k]\, x[n_0-k] \qquad \text{Non-Causal}$$

$\hat{x}[n_0]$ depends on past and future inputs.

Wiener Smoothing



Given $x[n] = s[n] + w[n]$, $-\infty < n < \infty$

$s[n] \rightarrow$ Signal

$w[n] \rightarrow$ Noise

Estimate $h[n]$ such that

$$\hat{S}[n_0] = \sum_{k=-\infty}^{+\infty} h[k]\; x[n_0-k]$$

$s[n], w[n] \longrightarrow$ WSS zero mean, known ACS (PSD)

ACS $r_s[l]$, $r_w[l]$

PSD $P_s(f)$, $P_w(f)$

$E[s[n]] = 0$

$E[w[n]] = 0$

$s[n]$ and $w[n]$ are uncorrelated.

$$E\big[s[n_1]\, w[n_2]\big] = 0 \qquad \forall\; n_1, n_2$$

Error $\epsilon[n_0] = s[n_0] - \hat{s}[n_0] = s[n_0] - \sum_{k=-\infty}^{+\infty} h[k]\, x[n_0-k]$

Mean Squared Error (MSE)

$$mse = E\left[\epsilon^2[n_0]\right] = E\left[\left(s[n_0] - \hat{s}[n_0]\right)^2\right]$$

$$mse = E\Big[\big(\underbrace{s[n] - \sum_{k=-\infty}^{+\infty} h[k]\, x[n_0-k]}_{\hat{s}[n_0]}\big)^2\Big]$$

Similar to
linear prediction

$$\frac{\partial mse}{\partial h[l]} = 0 \qquad -\infty < l < \infty \qquad \text{fix } l$$

$$E\Big[-2\big(s[n_0] - \sum_{k=-\infty}^{+\infty} h[k]\, x[n_0-k]\big)\, x[n_0-l]\Big] = 0$$

$$E\big[\,\epsilon[n_0]\; x[n_0-l]\,\big] = 0$$

Orthogonality Principle
$-\infty < l < \infty$

$$E\Big[\big(s[n_0] - \sum_{k=-\infty}^{+\infty} h[k]\, x[n_0-k]\big)\, x[n_0-l]\Big] = 0$$

$$E\big[s[n_0]\, x[n_0-l]\big] = \sum_{k=-\infty}^{+\infty} h[k]\; E\big[x[n_0-k]\, x[n_0-l]\big]$$

①

LHS

$$E\big[s[n_0]\, x[n_0-l]\big] = E\Big[s[n_0]\big(s[n_0-l] + w[n_0-l]\big)\Big]$$

$$= E\big[s[n_0]\, s[n_0-l]\big] + \underset{0}{\underset{\shortparallel}{E\big[s[n_0]\, w[n_0-l]\big]}}$$

$$= E\big[s[n_0]\, s[n_0-l]\big] = r_s[n_0-l-n_0]$$

$$= r_s[-l]$$

$$= r_s[l] \quad \text{———} \; ②$$

RHS

$$E\left[x[n_0-k]\,x[n_0-l]\right] = E\left[\left(s[n_0-k]+w[n_0-k]\right)\left(s[n_0-l]+w[n_0-l]\right)\right]$$

as $E\left[s[n_0-k]\,w[n_0-l]\right]=0$

$E\left[s[n_0-l]\,w[n_0-k]\right]=0$

$$= E\left[s[n_0-k]\,s[n_0-l]\right] + E\left[w[n_0-k]\,w[n_0-l]\right]$$

② and ③ in ①

$$= r_s[l-k] + r_w[l-k] \quad \text{③}$$

$$\Rightarrow \quad r_s[l] = \sum_{k=-\infty}^{+\infty} h[k]\left(r_s[l-k]+r_w[l-k]\right) \qquad -\infty < l < \infty$$

$$r_s[l] = h_{opt}[l] * \left(r_s[l]+r_w[l]\right)$$

DTFT

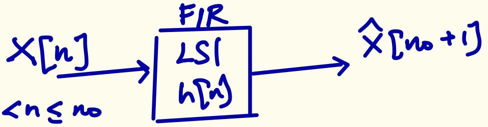
$$\Rightarrow \quad P_s(f) = H_{opt}(f)\left(P_s(f)+P_w(f)\right)$$

$$H_{opt}(f) = \frac{P_s(f)}{P_s(f)+P_w(f)}$$

$$h_{opt}[n] = \mathcal{F}^{-1}\left(H_{opt}(f)\right) = \underbrace{\int_{-\frac{1}{2}}^{\frac{1}{2}} H_{opt}(f)\, e^{-j2\pi nf}\, df}_{\text{IDTFT}}$$

$$\hat{s}[n_0] = \sum_{k=-\infty}^{+\infty} h_{opt}[k]\, x[n_0-k]$$

Wiener Prediction



$$\hat{x}[n_0+1] = \sum_{k=0}^{\infty} h[k]\, x[n_0-k] \quad , \quad e[n] = x[n_0+1] - \hat{x}[n_0+1]$$

$$mse = E\left[\left(x[n_0+1] - \hat{x}[n_0+1]\right)^2\right]$$

$$= E\left[\left(x[n_0+1] - \sum_{k=0}^{\infty} h[k]\, x[n_0-k]\right)^2\right]$$

$$\frac{\partial\, mse}{\partial h[l]} = 0$$

$$\Rightarrow E\left[\left(x[n_0+1] - \sum_{k=0}^{\infty} h[k]\, x[n_0-k]\right) x[n_0-l]\right] = 0$$

$$l = 0, 1, 2, \cdots$$

$$E\left[x[n_0+1]\, x[n_0-l]\right] = \sum_{k=0}^{\infty} h[k]\, E\left[x[n_0-k]\, x[n_0-l]\right]$$

$$r_x[l+1] = \sum_{k=0}^{\infty} h[k]\, r_x[l-k] \qquad l = 0, 1, 2, \cdots$$

$$l=0 \quad r_x[1] = \sum_{k=0}^{\infty} h[k]\, r_x[-k]$$

$$l=1 \quad r_x[2] = \sum_{k=0}^{\infty} h[k]\, r_x[1-k]$$

$$l=M-1 \quad r_x[M] = \vdots \ \sum_{k=0}^{\infty} h[k]\, r_x[M-1-k]$$

Solve the simultaneous linear equations.

For FIR with length M $\quad h[n] = 0 \quad$ for $n<0$ and $n \geq M$

$$r_x[l+1] = \sum_{k=0}^{M-1} h[k]\, r_x[l-k] \qquad l = 0, 1, 2, \ldots, M-1$$

Solve the linear system of equations.

$$\underbrace{\begin{bmatrix} r_x[0] & r_x[-1] & \cdots & r_x[-(M-1)] \\ r_x[1] & r_x[0] & \cdots & r_x[-(M-2)] \\ \vdots & & \ddots & \vdots \\ r_x[M-1] & & \cdots & r_x[0] \end{bmatrix}}_{\substack{M\times M \\ A}} \underbrace{\begin{bmatrix} h[0] \\ h[1] \\ \vdots \\ h[M-1] \end{bmatrix}}_{\substack{M\times 1 \\ \vec{h}_{opt}}} = \underbrace{\begin{bmatrix} r_x[1] \\ r_x[2] \\ \vdots \\ r_x[M] \end{bmatrix}}_{\substack{M\times 1 \\ \vec{b}}}$$

$$\underset{\underset{h[k]}{\uparrow}}{\vec{h}_{opt}} = A^{-1}\vec{b}$$

$$\hat{x}[n_0+1] = \sum_{k=0}^{M-1} h_{opt}[k]\, x[n_0-k]$$

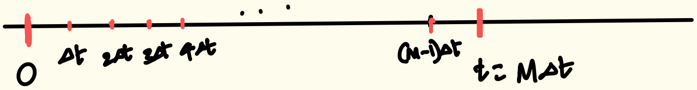
Poisson Random Processes

- Poisson RP $X(t) \to$ CTDV
  - Arrival Times
  - Bernoulli RP on real line
- Poisson Counting RP $N(t) \to$ CTDV
  - Number of Arrivals in $[0,t]$
  - Semi-infinite RP
  - Binomial RP on real line

The interval $[0,t]$ is split into M intervals of length Δt each.

$$t = M\Delta t$$

as $\Delta t \to 0$, $M \to \infty$



- One arrival can be in every slot of length Δt.
  - 0 — No arrival.
  - 1 — Arrival

Number of arrivals in $[0, t]$ equal to k

$$N(t) = k$$

Using Binomial PMF

$$P[N(t) = k] = \binom{M}{k} p^k (1-p)^{M-k}$$

$$Mp = E[N(t)] \longrightarrow \text{Expectation of Binomial}$$

As $\Delta t \to 0$, $M \to \infty$

$$E[N(t)] = Mp \to \text{finite}$$

$$N(t) \sim \text{Poisson}(\lambda')$$

$$\lambda' = E[N(t)] = Mp \longrightarrow \text{Expectation of Poisson}$$

$$P[N(t) = k] = \frac{e^{-\lambda'} (\lambda')^k}{k!} \text{———} ①$$

$$= \frac{e^{-E[N(t)]} (E[N(t)])^k}{k!}, \quad k = 0, 1, 2, \ldots$$

$$\lambda' = E[N(t)] = \lim_{\substack{M\to\infty \\ p\to 0}} Mp$$

$$= \lim_{\substack{\Delta t\to 0 \\ p\to 0}} \left(\frac{t}{\Delta t}\right) p$$

$$= t \lim_{\substack{\Delta t\to 0 \\ p\to 0}} \frac{p}{\Delta t}$$

$$= \lambda t \qquad \text{where } \lambda = \lim_{\substack{\Delta t\to 0 \\ p\to 0}} \frac{p}{\Delta t} \quad ②$$

② in ①

$$P[N(t)=k] = \frac{e^{-\lambda t}(\lambda t)^k}{k!}, \quad k=0,1,2,\dots$$

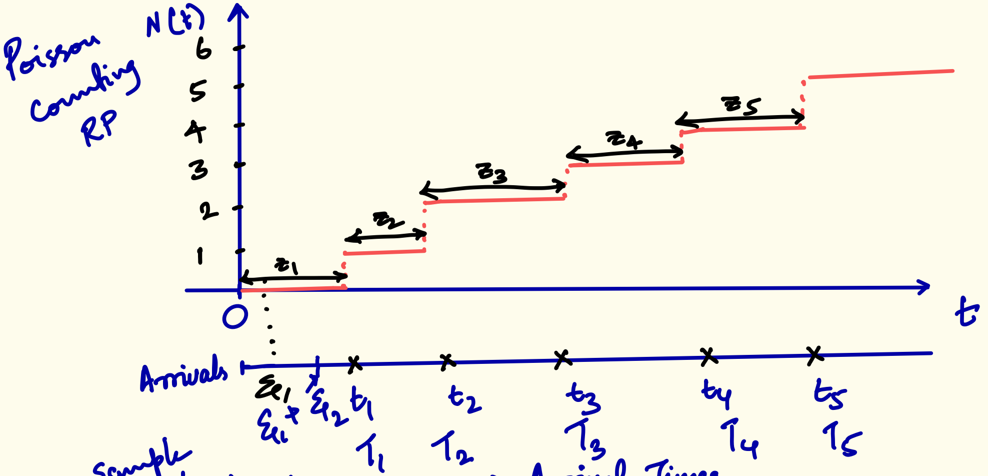
$N(t)$ → Poisson Counting RP

$N(0) = 0$, No arrivals at start.

$N(t)$ → Not SSS and Not WSS

as $E[N(t)] = \lambda t$ → Expectation varies with time t.

Inter-arrival Times



sample $t_1, t_2, t_3, \dots$ → Arrival Times

RVs $T_1, T_2, T_3, \dots$ Arrival Time RP

sample $z_1, z_2, z_3, \dots$ → Interarrival Times

RVs $Z_1, Z_2, Z_3, \dots$ Inter Arrival time RP

$Z_1 = T_1$ first arrival

If $Z_1 > \varepsilon_1$, $N(\varepsilon_1) = 0$

If $N(\varepsilon_1) = 0$, $z_1 > \varepsilon_1$

$$P[z_1 > \varepsilon_1] = P[N(\varepsilon_1) = 0] = \frac{e^{-\lambda \varepsilon_1} (\lambda \varepsilon_1)^0}{0!}$$

$$= e^{-\lambda \varepsilon_1}, \quad \varepsilon_1 \geq 0$$

$$P[z_1 < \varepsilon_1] = 1 - e^{-\lambda \varepsilon_1}$$

PDF for first arrival

$$p_{Z_1}(z_1) = \frac{d}{dz_1}\left[1 - P(Z_1 > z_1)\right]$$

$$= \frac{d}{dz_1}\left[1 - e^{-\lambda z_1}\right]$$

$$= \lambda e^{-\lambda z_1}$$

$$p_{Z_1}(z_1) = \begin{cases} \lambda e^{-\lambda z_1}, & z_1 \geq 0 \\ 0, & z_1 < 0 \end{cases}$$ Exponential PDF

$$Z_1 \sim \exp(\lambda)$$

Waiting for an arrival

Probability of arrival time $(Z_1) > \xi_1 + \xi_2$

given that $Z_1 > \xi_1$

No arrivals in $[0, \xi_1]$, $N(\xi_1) = 0$

$P[$ No arrivals in $[0, \xi_1 + \xi_2]]$

$N(\xi_1 + \xi_2) = 0$

$$P\left[z_1 > \varepsilon_{e_1} + \varepsilon_{e_2} / z_1 > \varepsilon_{e_1}\right] = \frac{P[z_1 > \varepsilon_{e_1} + \varepsilon_{e_2}, z_1 > \varepsilon_{e_1}]}{P[z_1 > \varepsilon_{e_1}]}$$

$$= \frac{P[z_1 > \varepsilon_{e_1} + \varepsilon_{e_2}]}{P[z_1 > \varepsilon_{e_1}]}$$

$$= \frac{e^{-\lambda(\varepsilon_{e_1} + \varepsilon_{e_2})}}{e^{-\lambda \varepsilon_{e_1}}}$$

$$= e^{-\lambda \varepsilon_{e_2}}$$

$$P\left[z_1 > \varepsilon_{e_1} + \varepsilon_{e_2} / z_1 > \varepsilon_{e_1}\right] = P\left[z_1 > \varepsilon_{e_2}\right]$$

$\Rightarrow$ Waiting for ε_{e_1} seconds not taken care.

Memoryless Property.

In general, Inter arrival times

$$Z_1, Z_2, \dots, Z_k \text{ are IID} \rightarrow \text{SSS RP}$$

$$Z_i \sim \exp(\lambda) \quad , \; i = 1, 2, \dots, k$$

Poisson RP → limiting case of Bernoulli RP

Bernoulli RP $\{X[0]=0, X[1], X[2], \dots\}$

Let two inter-arrival times $k_1 \geq 1$, $k_2 \geq 1$

eg. if $X[0]=0$, $X[1]=0$, $X[2]=1$, $X[3]=0$, $X[4]=0$, $X[5]=1$

$\Rightarrow$ $k_1 = 2$, $k_2 = 3$

$$P[Z_1 = k_1 , Z_2 = k_2]$$

$$= P\Big[\, X[n]=0 \text{ for } 1 \leq n \leq k_1 - 1, \; X[k_1] = 1, \; X[n] = 0 \text{ for } k_1+1 \leq n \leq k_1+k_2-1, \; X[k_1+k_2] = 1 \Big]$$

Geometric PMF

$$= \left[(1-p)^{k_1-1} p\right] \left[(1-p)^{k_2-1} p\right]$$

$$= P[Z_1 = k_1] \, P[Z_2 = k_2]$$

$\Rightarrow$ Joint PMF → factors into two marginal geometric PMFs

If $k_1 = k_2 \Rightarrow z_1, z_2$ are IID

Arrival Time RP

$z_1 = T_1 \qquad z_1 \sim exp(\lambda)$

$E[z_1] = E[T_1] = \frac{1}{\lambda}$

T_k → k^{th} arrival time

→ time from $t=0$ to k^{th} arrival

$$T_k = \sum_{i=1}^{k} z_i$$

z_i's are IID

$z_i \sim exp(\lambda)$

$$\Phi_{T_k}(\omega) = \left(\Phi_z(\omega)\right)^k$$

$$p_{T_k}(t) = \underbrace{p_z(\omega) * p_z(\omega) * \cdots * p_z(\omega)}_{k-1 \text{ convolutions}}$$

$$\Phi_z(\omega) = \int_0^\infty \lambda e^{-\lambda z} e^{j\omega z} dz = \lambda \int_0^\infty e^{-(\lambda - j\omega) z} dz$$

$$\Phi_z(\omega) = \lambda \left. \frac{e^{-(\lambda - j\omega)z}}{-(\lambda - j\omega)} \right|_0^{\infty}$$

$$= \frac{\lambda}{\lambda - j\omega}$$

$$\Phi_{T_k}(\omega) = \left(\Phi_z(\omega) \right)^k = \left(\frac{\lambda}{\lambda - j\omega} \right)^k$$

$$p_{T_k}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \Phi_{T_k}(\omega) \, e^{-j\omega t} \, d\omega$$

$$T_k \sim \Gamma(k, \lambda)$$

$$p_{T_k}(t) = \frac{\lambda^k}{(k-1)!} t^{k-1} e^{-\lambda t}$$ → Erlang PDF

$$E[T_k] = \frac{k}{\lambda} = k \, E[T_1]$$

Markov Chains

Consider a DTDV RP X[n] which takes only 2 possible values at every n.

Number of states $K = 2$

X[n] is a Markov chain if Conditional PMF

$$p_{X[n]/X[n-1],X[n-2],\dots,X[0]} = p_{X[n]/X[n-1]} \rightarrow \begin{array}{l}\text{First}\\ \text{order}\end{array}$$

We know that the joint PMF

$$p_{X[0],X[1],X[2],\dots,X[n]} = p_{X[n]/X[n-1]\dots X[0]}\; p_{X[n-1]/X[n-2]\dots X[0]} \cdots p_{X[1]/X[0]}\; p_{X[0]}$$

Using Markov property

$$\underbrace{p_{X[0],X[1],\dots,X[n]}}_{\text{Joint Prob.}} = \underbrace{p_{X[n]/X[n-1]}\; p_{X[n-1]/X[n-2]} \cdots p_{X[1]/X[0]}}_{\substack{\text{1st order Conditional Prob.}\\ \text{(PMF)}}}\; \underset{\substack{\downarrow\\ \text{Initial}\\ \text{Prob.}}}{p_{X[0]}}$$

$P_{ij} = P\left[x[n]=j \,/\, x[n-1]=i \right]$ → state Transition probability

$i = 0, 1, 2, \ldots, K-1$

$j = 0, 1, 2, \ldots, K-1$

When $K = 2$

$i = 0, 1$

$j = 0, 1$

Transition Probability Matrix

$$P = \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix}$$

sum along each row = 1

$P_{00} = P\left[x[n]=0 \,/ x[n-1] = 0 \right]$

$P_{10} = P\left[x[n]=0 \,/\, x[n-1] = 1 \right]$

$P_{01} = P\left[x[n]=1 \,/\, x[n-1] = 0 \right]$

$P_{11} = P\left[x[n]=1 \,/\, x[n-1] = 1 \right]$

P → constant Matrix

State Probabilities at time n

$$p_i[n] = P[X[n] = i] \quad , \quad i = 0, 1, 2, \ldots, K-1$$

when $K=2 \rightarrow$ Two states

$$p_0[n] = P[X[n]=0]$$

$$p_1[n] = P[X[n]=1]$$

PMF of X[n] $\rightarrow \vec{p}[n] = \begin{bmatrix} p_0[n] \\ p_1[n] \end{bmatrix}$ $\quad p_0[n] + p_1[n] = 1$

PMF changes with n.

$\Longrightarrow$ Markov Chain is non-stationary.

<u>Definition</u> Markov Chain

$X[n] \rightarrow$ DTDV RP Semi-infinite $n = 0, 1, 2, \ldots$

Sample Space / States $\rightarrow k = 0, 1, 2, \ldots, K-1$

State Probability Vector $\rightarrow \vec{p}[n] = \begin{bmatrix} p_0[n] \\ p_1[n] \\ \vdots \\ p_{K-1}[n] \end{bmatrix}$

PMF of X[n]

$$p_k[n] = P[X[n] = k]$$

State Transition Probability Matrix (Conditional Probabilities)

$$P = \begin{bmatrix} P_{0,0} & P_{0,1} & \cdots & P_{0,K-1} \\ P_{1,0} & P_{1,1} & \cdots & P_{1,K-1} \\ \vdots & & \ddots & \\ P_{K-1,0} & P_{K-1,1} & \cdots & P_{K-1,K-1} \end{bmatrix}_{K \times K}$$

where $P_{i,j} = P\left[X[n]=j \,/\, X[n-1]=i \right]$

Initial State Probability Vector

(PMF of $X[0]$)

$$\vec{p}[0] = \begin{bmatrix} p_0[0] \\ p_1[0] \\ \vdots \\ p_{K-1}[0] \end{bmatrix}$$

Two-state Markov Chain

$$\underset{\text{Marginal}}{P\left[X[n]=j\right]} = \sum_{i=0}^{1} \underset{\text{Joint}}{P\left[X[n-1]=i,\ X[n]=j\right]}$$

$$= \sum_{i=0}^{1} \underset{\text{Conditional}}{P\left[X[n]=j \,/\, X[n-1]=i\right]} \ \underset{\text{Marginal}}{P\left[X[n-1]=i\right]}$$

$$p_j[n] = \sum_{i=0}^{1} P_{ij} \ p_i[n-1] \ , \quad j=0,1$$

$$\underbrace{\begin{bmatrix} p_0[n] & p_1[n] \end{bmatrix}}_{\vec{p}^T[n] \quad \text{PMF of } X[n]} = \underbrace{\begin{bmatrix} p_0[n-1] & p_1[n-1] \end{bmatrix}}_{\vec{p}^T[n-1] \quad \text{PMF of } X[n-1]} \underbrace{\begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix}}_{P}$$

$$\underset{\text{PMF of } X[n]}{\vec{p}^{\,T}[n]} = \underset{\text{PMF of } X[n-1]}{\vec{p}^{\,T}[n-1]} \; P$$

eg: Let

$$P = \begin{bmatrix} 3/4 & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\vec{p}^{\,T}[0] = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \longrightarrow \text{PMF of } X[0]$$

PMF of X[1]

$$\vec{p}^{\,T}[1] = \vec{p}^{\,T}[0] \; P$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{8} & \frac{3}{8} \end{bmatrix}$$

PMF of X[2]

$$\vec{p}^{\,T}[2] = \vec{p}^{\,T}[1] \; P$$

$$= \begin{bmatrix} \frac{5}{8} & \frac{3}{8} \end{bmatrix} \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{21}{32} & \frac{11}{32} \end{bmatrix}$$

In general

$$\vec{p}^{\,T}[n] = \vec{p}^{\,T}[n-1] \; P$$

State Probabilities

$\overset{\text{PMF}}{\vec{p}[n_1]}$ to $\overset{\text{PMF}}{\vec{p}[n_2]} \rightarrow$ How?

Let $n_2 = n_1 + 2 \Rightarrow n_1 = n_2 - 2$

$$\vec{p}^T[n_2] = \vec{p}^T[n_2-1]\,P$$

$$= \left(\vec{p}^T[n_2-2]\,P\right)P$$

$$= \vec{p}^T[n_2-2]\,P^2$$

$$= \vec{p}^T[n_1]\,P^2$$

$P^2 \rightarrow$ Two-step Transition Probability Matrix.

In general,

$$\vec{p}^T[n_1+n] = \vec{p}^T[n_1]\,P^n$$

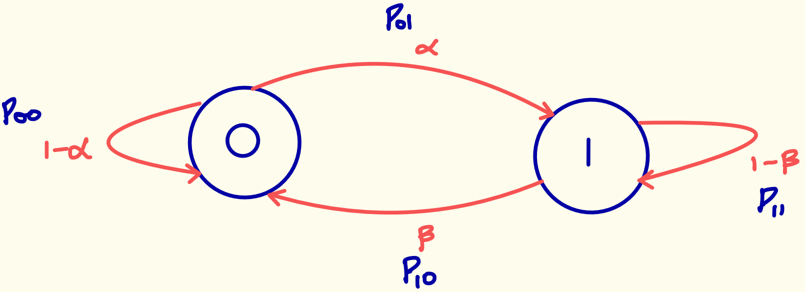
$P^n \rightarrow$ n-step Transition Probability Matrix.

If $n_1 = 0$

PMF of $x[n]$ $\quad \vec{p}^T[n] = \vec{p}^T[0]\,P^n$

$\vec{p}^T[0] \rightarrow$ Initial state Probability Vector.
PMF of $x[0]$

Two-state Probability → Markov chain



$$P = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix} \qquad \begin{array}{l} 0 \le \alpha \le 1 \\ 0 \le \beta \le 1 \end{array}$$

eg. Let $\alpha = \beta = \frac{1}{2}$ $\quad P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$, $P^n = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$, $n \ge 1$

PMF of $X[0]$ $\quad \vec{p}[0] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

PMF of $X[1]$ $\quad \vec{p}^T[1] = \vec{p}^T[0]\, P = \left[\frac{1}{2} \;\; \frac{1}{2}\right]$

PMF of $X[2]$ $\quad \vec{p}^T[2] = \vec{p}^T[0]\, P^2 = \left[\frac{1}{2} \;\; \frac{1}{2}\right]$

$\vdots \qquad\qquad \vdots$

PMF of $X[n]$ $\quad \vec{p}^T[n] = \vec{p}^T[0]\, P^n = \left[\frac{1}{2} \;\; \frac{1}{2}\right]$, $\forall\, n \ge 1$

steady state PMF

Markov chain is in steady state.

→ Depends on the form of P.

Powers of P

Let P has distinct eigenvalues, λ_1 and λ_2.

The eigen vectors $\vec{v}_i$ → linearly Independent.

$$V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \qquad \text{rank}(V) = 2$$

We know that

$$P = V \Lambda V^{-1}$$

Eigenvalue Decomposition

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

$$P^2 = V \Lambda V^{-1} V \Lambda V^{-1} = V \Lambda^2 V^{-1}$$

$$P^3 = V \Lambda^3 V^{-1}$$

$$\vdots$$

$$P^n = V \Lambda^n V^{-1}$$

where $\Lambda^n = \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix}$

$$P^n = V \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} V^{-1}$$

eg. $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$

$$\det(P - \lambda I) = 0 \Rightarrow \det \begin{bmatrix} \frac{1}{2} - \lambda & \frac{1}{2} \\ 0 & 1-\lambda \end{bmatrix} = 0$$

$$\left(\tfrac{1}{2} - \lambda\right)(1-\lambda) = 0$$

$$\Rightarrow \lambda_1 = \tfrac{1}{2}, \quad \lambda_2 = 1$$

$$(P - \overset{\frac{1}{2}}{\lambda_1} I)\vec{v}_1 = \vec{0} \Rightarrow \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix} \vec{v}_1 = \vec{0} \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$(P - \overset{1}{\lambda_2} I)\vec{v}_2 = \vec{0} \Rightarrow \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \vec{v}_2 = \vec{0} \Rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$V = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad V^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

For $n \geq 1$

$$P^n = V \Lambda^n V^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (\frac{1}{2})^n & 0 \\ 0 & 1^n \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (\frac{1}{2})^n & -(\frac{1}{2})^n \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} (\frac{1}{2})^n & 1 - (\frac{1}{2})^n \\ 0 & -1 \end{bmatrix}$$

Two state Markov Chain

$$P = \begin{bmatrix} \overset{P_{00}}{1-\alpha} & \overset{P_{01}}{\alpha} \\ \overset{P_{10}}{\beta} & \overset{P_{11}}{1-\beta} \end{bmatrix} \longrightarrow$$ State Transition Probability Matrix

$$\det(P-\lambda I) = \det\begin{bmatrix} 1-\alpha-\lambda & \alpha \\ \beta & 1-\beta-\lambda \end{bmatrix} = 0$$

$$\Rightarrow (1-\alpha-\lambda)(1-\beta-\lambda) - \alpha\beta = 0$$

$$\Rightarrow \lambda^2 + (\alpha+\beta-2)\lambda + (1-\alpha-\beta) = 0$$

Let $\gamma = \alpha+\beta$

$$\Rightarrow \lambda^2 + (\gamma-2)\lambda + (1-\gamma) = 0$$

$$\lambda = \frac{-(\gamma-2) \pm \sqrt{(\gamma-2)^2 - 4(1-\gamma)}}{2}$$

$$= \frac{-(\gamma-2) \pm \sqrt{\gamma^2 \cancel{+4} - \cancel{4\gamma} - \cancel{4} + \cancel{4\gamma}}}{2}$$

$$= \frac{-(\gamma-2) \pm \gamma}{2}$$

$$\lambda_1 = \frac{-(\gamma-2)+\gamma}{2} = 1$$

$$\lambda_2 = \frac{-(\gamma-2)-\gamma}{2} = \frac{-2\gamma+2}{2} = 1-\gamma = 1-\alpha-\beta$$

eigenvalues of P

$\lambda_1 = 1$

$\lambda_2 = 1-\alpha-\beta$

$$P = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$$

$$(P - \overset{1}{\lambda_1} I)\vec{v_1} = \vec{0} \Rightarrow \begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix} \vec{v_1} = \vec{0}$$

$$\Rightarrow \vec{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$(P - \lambda_2 I)\vec{v_2} = \vec{0} \Rightarrow \begin{bmatrix} \beta & \alpha \\ \beta & \alpha \end{bmatrix} \vec{v_2} = \vec{0}$$

$$\Rightarrow \vec{v_2} = \begin{bmatrix} 1 \\ -\beta/\alpha \end{bmatrix}$$

$$V = \begin{bmatrix} 1 & 1 \\ 1 & -\beta/\alpha \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 1-\alpha-\beta \end{bmatrix}$$

$$V^{-1} = -\frac{1}{(1+\beta/\alpha)} \begin{bmatrix} -\beta/\alpha & -1 \\ -1 & 1 \end{bmatrix}$$

$$\Lambda^n = \begin{bmatrix} 1 & 0 \\ 0 & (1-\alpha-\beta)^n \end{bmatrix}$$

$$P^n = V \Lambda^n V^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -\beta/\alpha \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (1-\alpha-\beta)^n \end{bmatrix} \left(-\frac{1}{(1+\beta/\alpha)} \right) \begin{bmatrix} -\beta/\alpha & -1 \\ -1 & 1 \end{bmatrix}$$

$$= -\frac{1}{(1+\beta/\alpha)} \left[1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} -\beta/\alpha & -1 \end{bmatrix} + (1-\alpha-\beta)^n \begin{bmatrix} 1 \\ -\beta/\alpha \end{bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix} \right]$$

$$= -\frac{1}{(1+\beta/\alpha)} \left[\begin{bmatrix} -\beta/\alpha & -1 \\ -\beta/\alpha & -1 \end{bmatrix} + (1-\alpha-\beta)^n \begin{bmatrix} -1 & 1 \\ +\beta/\alpha & -\beta/\alpha \end{bmatrix} \right]$$

$$= \begin{bmatrix} \frac{\beta/\alpha}{1+\beta/\alpha} & \frac{1}{1+\beta/\alpha} \\ \frac{\beta/\alpha}{1+\beta/\alpha} & \frac{1}{1+\beta/\alpha} \end{bmatrix} + (1-\alpha-\beta)^n \begin{bmatrix} \frac{1}{1+\beta/\alpha} & -\frac{1}{1+\beta/\alpha} \\ \frac{-\beta/\alpha}{1+\beta/\alpha} & \frac{\beta/\alpha}{1+\beta/\alpha} \end{bmatrix} \quad \text{—①}$$

$$\frac{\beta/\alpha}{1+\beta/\alpha} = \frac{\alpha(\beta/\alpha)}{\alpha+\beta} = \frac{\beta}{\alpha+\beta} \quad \text{—②}$$

$$\frac{1}{1+\frac{\beta}{\alpha}} = \frac{\alpha}{\alpha+\beta} \quad \text{—③}$$

② & ③ in ①

$$\Rightarrow P^n = \begin{bmatrix} \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \\ \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \end{bmatrix} + (1-\alpha-\beta)^n \begin{bmatrix} \frac{\alpha}{\alpha+\beta} & -\frac{\alpha}{\alpha+\beta} \\ -\frac{\beta}{\alpha+\beta} & \frac{\beta}{\alpha+\beta} \end{bmatrix}$$

as $0 \leq \alpha \leq 1$ and $0 \leq \beta \leq 1$,

$$0 \leq \alpha+\beta \leq 2$$

As $\lambda_2 = 1-\alpha-\beta = 1-(\alpha+\beta)$,

$$-1 \leq \lambda_2 \leq 1 \quad \text{or} \quad -1 \leq 1-\alpha-\beta \leq 1$$

Case 1 $\quad -1 < \lambda_2 = 1-\alpha-\beta < 1 \quad \Rightarrow |1-\alpha-\beta| < 1$

$$\Rightarrow \lim_{n\to\infty} (1-\alpha-\beta)^n \longrightarrow 0$$

$$\lim_{n\to\infty} P^n = \lim_{n\to\infty} \begin{bmatrix} \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \\ \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \end{bmatrix} + \underbrace{\lim_{n\to\infty} (1-\alpha-\beta)^n}_{0} \begin{bmatrix} \frac{\alpha}{\alpha+\beta} & -\frac{\alpha}{\alpha+\beta} \\ -\frac{\beta}{\alpha+\beta} & \frac{\beta}{\alpha+\beta} \end{bmatrix}$$

$$P^n \longrightarrow \begin{bmatrix} \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \\ \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \end{bmatrix} \quad \text{as } n\to\infty$$

$$\vec{p}^{\,T}[n] = \vec{p}^{\,T}[0]\, P^n = \begin{bmatrix} p_0[0] & p_1[0] \end{bmatrix} \begin{bmatrix} \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \\ \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \end{bmatrix}$$

$$\vec{p}^{\,T}[n] = \begin{bmatrix} p_0[0]\frac{\beta}{\alpha+\beta} + p_1[0]\frac{\beta}{\alpha+\beta} \\ p_0[0]\frac{\alpha}{\alpha+\beta} + p_1[0]\frac{\alpha}{\alpha+\beta} \end{bmatrix}^T \qquad p_0[0] + p_1[0] = 1$$

$$= \begin{bmatrix} \frac{\beta}{\alpha+\beta} \\ \frac{\alpha}{\alpha+\beta} \end{bmatrix}^T$$

$$\Rightarrow \begin{bmatrix} p_0[n] \\ p_1[n] \end{bmatrix} = \begin{bmatrix} \frac{\beta}{\alpha+\beta} \\ \frac{\alpha}{\alpha+\beta} \end{bmatrix}$$ → steady state irrespective of $\vec{p}^{\,T}[0]$.

Ergodic Markov chain.

$\vec{p}^{\,T}[\infty] \longrightarrow \vec{\pi}^{\,T} = [\pi_0 \ \ \pi_1] = \left[\frac{\beta}{\alpha+\beta} \ \ \frac{\alpha}{\alpha+\beta}\right]$ → Steady state

All rows of P^n same as $n \to \infty$.

<u>Case 2</u> $\alpha = \beta = 0$, $1-\alpha-\beta = 1$ $\Rightarrow$ $\lambda_1 = \lambda_2 = 1$

$\Rightarrow \Lambda^n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_{2\times 2}$ $\qquad P^n = V I V^{-1}$ for all n

$= V V^{-1}$

$= I$

$\Rightarrow \vec{p}^{\,T}[n] = \vec{p}^{\,T}[0]\,P^n = \vec{p}^{\,T}[0]$ → All PMFs are same

$\vec{\pi}^{\,T} = \vec{p}^{\,T}[0]$ → steady state

State diagram

Realizations

000 . . .

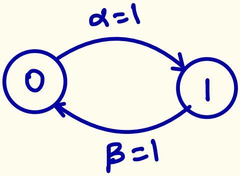
111 . . .

Case 3 $\lambda_2 = 1-\alpha-\beta = -1$ or $(\alpha=1 \text{ and } \beta=1)$

Realizations

0101 . . .

1010 . . .



$$P^n = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} + (-1)^n \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

For $n \to$ odd

$$P^n = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

For $n \to$ even

$$P^n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\vec{p}^T[n] = \vec{p}^T[0]\, P^n = [p_0[0] \;\; p_1[0]]\, P^n$$

$$= \begin{cases} [p_0[0] \;\; p_1[0]], & n \text{ even} \\ [p_1[0] \;\; p_0[0]], & n \text{ odd} \end{cases}$$

No steady state.